Your name: ______________________________

Math 1031 Practice Exam 1
October 2004

There are nine questions, worth varying percentages as shown. Show your work in the space provided. You may not use your books or notes or a graphing calculator on this exam. You may use a regular scientific calculator.

Formulas:
Fundamental Counting Principle: the number of ways to perform independent tasks $T_1, \ldots, T_k$ where there are n_i ways to perform T_i is the product $n_1 \cdots n_k$.

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n \cdot (n-1)!$$

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

$$C(n,k) = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 3 \cdot 2 \cdot 1} = \frac{n!}{(n-k)!\,k!} = C(n,n-k)$$

Probability
A sample space S consists of outcomes $s_1, \ldots, s_n$. Each outcome s_i is assigned a probability p_i with

$$0 \le p_i \le 1 \quad \text{and} \quad p_1 + \cdots + p_n = 1.$$

The probability of an event E is the sum of the probabilities of the outcomes in E. When all the outcomes are equally likely $p_i = \frac{1}{n}$ and:

$$P(E) = \frac{|E|}{|S|}.$$

It is always true that

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

If E and F are *mutually exclusive* then $P(E \cup F) = P(E) + P(F)$.
Events E and F are *independent* if $P(E \cap F) = P(E)P(F)$.
Also $P(E) + P(E^c) = 1$.
In independent experiments where $P(\text{success}) = p$ and $P(\text{failure}) = 1 - p$ we have

$$P(k \text{ successes in } n \text{ experiments}) = C(n,k)p^k(1-p)^{n-k}.$$

Conditional probability: $P(E|F) = \frac{P(E \cap F)}{P(F)}$.

1. (10%) How many 4-tuples are there whose entries are the letters A, B, C in which there is exactly one A?
   Solution:

   Use the fundamental counting principle.
   Task 1: choose a place for A, there are $C(4,1) = 4$ ways.
   Taks 2: put B and C at the rest 3 places, there are $2^3 = 8$ ways.
   Thus there are $4 * 8 = 32$ 4-tuples are there whose entries are the letters A, B, C in which there is exactly one A.

2. (12%) A committee of 4 men and 4 women is to be made from a group of people consisting of 6 men and 7 women
   (a) (6%) In how many ways can such a committee be made?
   Solution:

   Task 1: choose 4 men out of 6, there are $C(6,4) = 15$ ways.
   Task 2: choose 4 women out of 7, there are $C(7,4) = 35$ ways.
   Fundamental principle gives $15 * 35 = 525$ ways.
   (b) (6%) Suppose that one of the men is Mr. Smith and one of the women is his wife, Mrs. Smith. If Mr. and Mrs. Smith may not both serve on the same committee, in how many ways can such a committee be made?
   Solution:

   The number of ways that both Mr. and Mrs. Smith serve on the same committee is $C(5,3) * C(6,3) = 10 * 20 = 200$. Thus the numbe of ways that Mr. and Mrs. Smith may not both serve on the same committee is $525 - 200 = 325$.

3. (10%) I keep 5 different pairs of socks, which I wear when skiing, in a tall cardboard box. It is very difficult to see which sock you are pulling out of the box until you have taken it out. I draw out 2 socks at random from the total of 10. What is the probability that I have drawn out a matching pair?
(One way to start this is first to count how many pairs of socks (both matching and unmatching) there are altogether.)
Solution:

There are totally $C(10, 2) = 45$ pairs of socks, 5 of them are matching pairs, thus the probability that I have drawn out a matching pair is

$$\frac{5}{45} = \frac{1}{9}$$

.

4. (11%) You are dealt 5 cards from a standard deck of 52 cards. Find the probability of being dealt three of a kind; that is, having three cards of the same denomination, the remaining two cards being of two further denominations.
Solution:

The probability is

$$\frac{13 \cdot C(4, 3) \cdot 48 \cdot 44}{2 \cdot C(52, 5)}$$

.

5. (10%) The Security Council of the United Nations used to consist only of representatives of the five countries: USA, China, Russia, France and Britain. When they voted on a motion each country could either approve it, or veto it. If any country vetoed a motion, the motion failed. On one occasion it was known that each of the five countries was likely to approve a certain motion with probability 2/3, and to veto it with probability 1/3, independently of what the other countries do. Calculate the probability that the motion failed.

   Solution:

   Let E denote the event that the motion failed, then E^c is the event that the motion succeeded. Since each country must approve in order to succeed the motion, we have $P(E^c) = (\frac{2}{3})^5 = \frac{32}{243}$. Thus the probability that the motion failed is

$$P(E) = 1 - P(E^c) = 1 - \frac{32}{243} = \frac{211}{243}$$

   .

6. (12%) I throw a pair of dice, and am interested in the following events:

   A: the first die shows 2.
   B: the two dice show the same number.
   C: the total on the two dice is 5.

   For each pair of these events, say whether they are independent and/or mutually exclusive by writing 'Yes' or 'No' in each of the spaces provided below.

   (4%) Are A and B independent? $\underline{Yes}$ mutually exclusive? $\underline{No}$

   (4%) Are A and C independent? $\underline{No}$ mutually exclusive? $\underline{No}$

   (4%) Are B and C independent? $\underline{No}$ mutually exclusive? $\underline{Yes}$

7. (10%) Two dice are rolled. Find the probability that the two dice both show even numbers, given that the total on the two dice is 10.
Solution:

Let E be the event that both dice show even number, F be the event that the sum of two dice is 10. We have $F = \{(4,6), (5,5), (6,4)\}$ and $E \cap F = \{(4,6), (6,4)\}$. Hence

$$P(E|F) = \frac{|E \cap F|}{|F|} = \frac{2}{3}$$

.

8. (10%) You play a game in which you are dealt 2 cards from a deck of 52. If the cards are a pair (of the same denomination) you win \$1, and otherwise you pay 5¢. What is the expected value to you of this game?
Solution:

Let E be the event that the cards are a pair, then $P(E) = \frac{13 \cdot C(4,2)}{C(52,2)} = \frac{3}{51}$. The expected value of this game is

$$100 \cdot \frac{3}{51} - 5 \cdot \frac{48}{51} = \frac{60}{51} = 1.19¢$$

.

9. (15%) In a certain country where 10% of the population actually has hepatitis, a test is used which tests positive for hepatitis 90% of the time when a person has hepatitis, and which tests positive for hepatitis 30% of the time when the person does not have hepatitis.

   (a) (5%) Find the probability that a randomly chosen member of the population will test positive for hepatitis.

   Solution:

   Let E be the event that a person tests positive for hepatitis, and F be the event that a person has hepatitis. We know that $P(F) = 0.1, P(E|F) = 0.9, P(E|F^c) = 0.3$. Thus the probability that a randomly chosen member of the population will test positive for hepatitis is

$$P(E) = P(E|F) \cdot P(F) + P(E|F^c) \cdot P(F^c) = 0.9 \times 0.1 + 0.3 \times 0.9 = 0.36$$

   .

   (b) (5%) Find the probability that a randomly chosen member of the population both tests positive for hepatitis and also has hepatitis.

   Solution:

$$P(E \cap F) = P(E|F) \cdot P(F) = 0.9 \times 0.1 = 0.09$$

.

   (c) (5%) Find the probability that a person who tests positive for hepatitis actually has hepatitis.

   Solution:

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{0.09}{0.36} = \frac{1}{4}$$

.