Math 8246 **Homework 3** **PJW**

Date due: Monday March 5, 2007

1. Let V and W be vector spaces over $\mathbb{R}$. Given vector space endomorphisms $\alpha : V \to V$ and $\beta : W \to W$ we may make V and W into $\mathbb{R}[X]$-modules by defining $X \cdot v = \alpha(v)$, $X \cdot w = \beta(w)$ for $v \in V$ and $w \in W$. In each of the following cases where we specify α, β by means of matrices with respect to some bases, compute $\dim_{\mathbb{R}} \mathrm{Ext}^1_{\mathbb{R}[X]}(V, W)$.

   (i) $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \beta = (1)$

   (ii) $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

   (iii) $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

   (iv) $\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

   In case (i) above, exhibit a non-split extension of V by W (i.e. a non-split short exact sequence $0 \to W \to M \to V \to 0$ of $\mathbb{R}[X]$-modules).

2. Any endomorphism $\phi : N \to N$ of the R-module N gives rise to a homomorphism $\phi_* : \mathrm{Ext}^1_R(M, N) \to \mathrm{Ext}^1_R(M, N)$. (This makes $\mathrm{Ext}^1_R(M, N)$ into an $\mathrm{End}_R(N)$-module.) Let $0 \to N \to E \to M \to 0$ be an extension represented by $\xi \in \mathrm{Ext}^1_R(M, N)$. Consider commutative diagrams

$$\begin{array}{ccccccccccc} \xi: & 0 & \to & N & \to & E & \to & M & \to & 0 & \\ & & & \phi\big\downarrow & & \big\downarrow & & \big\| & & & \qquad (*) \\ & 0 & \to & N & \to & E_1 & \to & M & \to & 0 & \end{array}$$

   where the bottom row is again an extension.

   (i) Show that the bottom extension is represented by $\phi_*(\xi)$.

   (ii) Show that there exists a diagram $(*)$ if and only if the class in $\mathrm{Ext}^1_R(M, N)$ of the bottom extension lies in the $\mathrm{End}_R(N)$-submodule of $\mathrm{Ext}^1_R(M, N)$ generated by ξ.

   (iii) Consider $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Show that an extension $0 \to \mathbb{Z} \to E \to \mathbb{Z}/n\mathbb{Z} \to 0$ represented by $r + n\mathbb{Z}$ has the property that for every other extension there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & E & \to & \mathbb{Z}/n\mathbb{Z} & \to & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \to & \mathbb{Z} & \to & E_1 & \to & \mathbb{Z}/n\mathbb{Z} & \to & 0 \end{array}$$

   no matter what the lower extension is, if and only if r is prime to n.

3. Exhibit three inequivalent extensions $1 \to C_2 \to C_4 \times C_2 \to C_2 \times C_2 \to 1$.

4. Let $G = \langle g \rangle$ be an infinite cyclic group. Consider an extension of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z} \to 0$$

in which the maps are inclusion into the first summand and projection onto the second summand, and where g acts on $\mathbb{Z} \oplus \mathbb{Z}$ as the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with respect to the basis given by this direct sum decomposition. In the identification $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, determine the Ext class of this extension, and conclude that the extension is not split. Find a description of an extension represented by $5 \in \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$.

5. If N is a right $\mathbb{Z}G$-module and M is a left $\mathbb{Z}G$-module we may make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n \otimes m) = ng^{-1} \otimes gm$, extended linearly to the whole of $N \otimes_{\mathbb{Z}} M$. Show that $N \otimes_{\mathbb{Z}G} M \cong (N \otimes_{\mathbb{Z}} M)_G$.
[Not part of the question, just information: if N and M are two left modules we make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n \otimes m) = gn \otimes gm$. This is called the *diagonal action* on the tensor product.]

6. (a) Let M and N be $\mathbb{Z}G$-modules and suppose that N has the trivial G-action. Show that $\mathrm{Hom}_{\mathbb{Z}G}(M, N) \cong \mathrm{Hom}_{\mathbb{Z}G}(M/(IG \cdot M), N)$.
(b) Show that for all groups G, $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, IG) = 0$; and that if we suppose that G is finite then $\mathrm{Hom}_{\mathbb{Z}G}(IG, \mathbb{Z}) = 0$.
(c) By applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(IG, \quad)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ show that for all finite groups G, if $f : IG \to \mathbb{Z}G$ is any $\mathbb{Z}G$-module homomorphism then $f(IG) \subseteq IG$.
(d) Show that if G is finite and $d : G \to \mathbb{Z}G$ is any derivation then $d(G) \subseteq IG$. Is the same true for arbitrary groups G?

7. Show that for every group G:
(i) all derivations $d : G \to M$ satisfy $d(1) = 0$, and
(ii) the mapping $d : G \to \mathbb{Z}G$ given by $d(g) = g - 1$ is a derivation.

Further questions - do not hand in.

8. Let $0 \to \mathbb{Z} \to E \to \mathbb{Z}/n\mathbb{Z} \to 0$ be an extension of abelian groups represented by $r + n\mathbb{Z} \in \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ under the identification of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ with $\mathbb{Z}/n\mathbb{Z}$, where $r \in \mathbb{Z}$. Show that $E \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ where $d = h.c.f.\{r, n\}$ and identify the morphisms $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ in this extension, giving their components with respect to this direct sum decomposition of E.

9. (D&F 17.2 question 4) Suppose H is a normal subgroup of the group G and A is a $\mathbb{Z}G$-module. For every $g \in G$ prove that the map $f(a) = ga$ for $a \in A^H$ defines an automorphism of the subgroup A^H.

10. Let $0 \to N \to E \to M \to 0$ be an extension of modules over a ring R. Show that in the long exact Ext sequence

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,E) \to \mathrm{Hom}(M,M) \xrightarrow{\omega} \mathrm{Ext}^1(M,N) \to \cdots$$

the image of the identity 1_M under the connecting homomorphism ω is exactly the element of $\mathrm{Ext}^1(M,N)$ which represents the class of the extension.

Questions on the Free Differential Calculus of R.H. Fox

11. Suppose that G is generated by elements $g_1, \ldots, g_n$, that is $G = \langle g_1, \ldots, g_n \rangle$, and let $d : G \to M$ be a derivation. Show that for each element $g \in G$ there exist elements $\lambda_i \in \mathbb{Z}G$ (which will depend on g) such that $d(g) = \sum \lambda_i d(g_i)$. Conclude that the $\mathbb{Z}$-linear span of $d(G)$ is the $\mathbb{Z}G$-submodule of M generated by $d(g_1), \ldots, d(g_n)$.

12. Apply the last question in the case of the derivation $d : G \to \mathbb{Z}G$ given by $d(g) = g - 1$, so that this defines $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}G$. Show that now for any other derivation $e : G \to N$ we also have $e(g) = \sum \lambda_i e(g_i)$, this equation holding in N.

13. Let F be the free group on generators $g_1, \ldots, g_n$ and consider $d : F \to \mathbb{Z}F$ given by $d(g) = g - 1$. Show that the elements λ_i considered in the last questions are now uniquely determined. We will denote the element $\lambda_i \in \mathbb{Z}F$ by $\frac{\partial g}{\partial g_i}$. Thus

$$d(g) = \sum_i \frac{\partial g}{\partial g_i} d(g_i).$$

Show that
(i) the mapping $\frac{\partial}{\partial g_i} : F \to \mathbb{Z}F$ is a derivation,
(ii) $\frac{\partial g_j}{\partial g_i} = \delta_{ij}$.
The properties (i) and (ii) in fact characterize $\frac{\partial}{\partial g_i}$. Demonstrate this by computing $\frac{\partial}{\partial x}(yx^{-1}yx^2)$.
[The approach presented here is the 'free differential calculus' of R.H. Fox, and one of its uses can be found described in the book on knot theory by R.H. Crowell and R.H. Fox.]

14. Let $1 \to K \to E \to G \to 1$ be an extension of groups, with $K \leq E$. Show that if $k_1, \ldots, k_m$ generate K as a normal subgroup of E then $k_1K', \ldots, k_mK'$ generate K/K' as a $\mathbb{Z}G$-module.

15. Let $1 \to R \to F \to G \to 1$ be a presentation of G, so

$$G = \langle g_1 \ldots g_n \mid r_1, \ldots, r_m \rangle,$$

where R is the normal subgroup of F generated by $r_1, \ldots, r_m$. Show that

$$r_j - 1 = \sum_i \frac{\partial r_j}{\partial g_i}(g_i - 1), \quad j = 1, \ldots, m.$$

Deduce that in the start of the resolution

$$\begin{array}{ccccccccc} \mathbb{Z}G^m & & \xrightarrow{\alpha} & & \mathbb{Z}G^n & & \longrightarrow & & \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & R/R' & & & & IG & & \end{array}$$

where the basis vectors of $\mathbb{Z}G^m$ are mapped to the generators r_jR' of R/R', the map α has matrix $\left(\frac{\partial r_j}{\partial g_i}\right)_{i,j}$.
Evaluate this matrix in the case of the group given by the presentation

$$G = \langle x, y \mid x^4 = 1 = y^4,\, x^2 = y^2,\, yxy^{-1} = x^3 \rangle.$$

[One might call this the *Jacobian matrix*. It appears in the definition of the Alexander polynomial of a knot.]