Math 8246 **Homework 4** **PJW**
Date due: Monday March 26, 2007

1. Let $1 \to L \to J \to G \to 1$ be a short exact sequence of groups where L is an abelian subgroup of J and the mapping $L \to J$ is inclusion. In this situation, conjugation within J gives L the structure of a $\mathbb{Z}G$-module. Suppose that M is another $\mathbb{Z}G$-module and that $\theta : L \to M$ is a group homomorphism. Form the semidirect product $M \rtimes J$ and let $U = \{(-\theta(x), x) \mid x \in L\} \subseteq M \rtimes J$. (Here M is written additively and J acts on M via the homomorphism $J \to G$).
   (i) Show that U is a normal subgroup of $M \rtimes J$ if and only if θ is a $\mathbb{Z}G$-module homomorphism.
   (ii) Assuming that θ is a $\mathbb{Z}G$-module homomorphism, let $E = (M \rtimes J)/U$. Show that there is a commutative diagram of groups

$$\begin{array}{ccccccccc} 1 & \to & L & \xrightarrow{\gamma} & J & \to & G & \to & 1 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle 1_G} & & \\ 1 & \to & M & \xrightarrow{\alpha} & E & \to & G & \to & 1 \end{array}$$

   and that the bottom row is exact.

2. Suppose that we have two commutative diagrams of group homomorphisms

$$\begin{array}{ccccccccc} 1 & \to & L & \xrightarrow{\gamma} & J & \to & G & \to & 1 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi_i} & & \downarrow{\scriptstyle 1_G} & & \\ 1 & \to & M & \xrightarrow{\alpha_i} & E_i & \to & G & \to & 1 \end{array}$$

   where $i = 1, 2$, the maps labeled without the suffix i are the same in both diagrams, L and M are abelian and the two rows are group extensions (i.e. short exact sequences of groups). Show that the two bottom extensions are equivalent.

3. Given a free presentation $1 \to R \to F \to G \to 1$ of a group G we have constructed a short exact sequence of $\mathbb{Z}G$-modules $0 \to R/R' \to \mathbb{Z}G^{d(F)} \to IG \to 0$ where $d(F)$ is the rank of the free group F. By using this sequence, give a proof of the rank formula $d(R) = 1 + |G|(d(F) - 1)$ when G is finite. [Assume that R is a free group – subgroups of free groups are free.]

4. (D&F 17.3, 4) Let V be the Klein 4-group and let $G = \mathrm{Aut}(V) \cong S_3$ act on V in the natural fashion. Prove that $H^1(G, V) = 0$. [Show that in the semidirect product $E = V \rtimes G$, G is the normalizer of a Sylow 3-subgroup of E. Apply Sylow's Theorem to show all complements to V in E are conjugate.]

5. Let $1 \to M \xrightarrow{i} E \xrightarrow{p} G \to 1$ be a group extension in which M is abelian, and let $s : G \to E$ be a section for p, that is, a mapping which satisfies $ps = 1_G$. Thus each element of E can be written in the form $i(m) \cdot s(x)$ with m and x uniquely determined. The multiplication in E determines a function $f : G \times G \to M$ by

$$s(x) \cdot s(x') = if(x, x') \cdot s(xx'), \quad x, x' \in G.$$

(i) Show that associativity of multiplication in E implies

$$xf(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0, \quad \text{for all } x, y, z \in G.$$

A function f satisfying this condition is called a *factor set*.

(ii) Show that factor sets form a group under $(f_1 + f_2)(x, x') = f_1(x, x') + f_2(x, x')$.

(iii) Show that if $g : G \to M$ is any function then the function which sends (x, y) to $g(xy) - g(x) - xg(y)$ is a factor set.

(iv) Show that if $s, s' : G \to E$ are two sections and f, f' the corresponding factor sets, then there is a function $g : G \to M$ with $f'(x, y) = f(x, y) + g(xy) - g(x) - xg(y)$.

[In fact the quotient of the group of factor sets by the factor sets of the form g is isomorphic to $H^2(G, M)$ and we have gone some way towards showing from this point of view that this group bijects with equivalence classes of extensions.]

6. Let M be a normal subgroup of a group E, write $G = E/M$ and suppose that M is generated as a normal subgroup of E by elements $m_1, m_2, \ldots, m_s$.

(i) Show that $m_1M', \ldots, m_sM'$ generate M/M' as a $\mathbb{Z}G$-module.

(ii) Show that $m_1 - 1, \ldots, m_s - 1$ generate $\mathbb{Z}E \cdot IM = \mathbb{Z}E \cdot IM \cdot \mathbb{Z}E$ as a 2-sided ideal of $\mathbb{Z}E$.