Math 8246 **Homework 5** **PJW**

Date due: Monday April 9, 2007

1. Show that there are three equivalence classes of extensions

$$1 \to C_2 \to E \to C_2 \times C_2 \to 1$$

in which E is isomorphic to D_8 and one equivalence class in which E is isomorphic to Q_8. Show that the sum of the Q_8 extension with any of the D_8 extensions is an extension in which the middle group is abelian.

2. (a) Let G be a group with a presentation $G = \langle g_1, \ldots, g_d \mid a_1, \ldots, a_r \rangle$ and suppose that the abelianisation G/G' is the direct sum of a free abelian group of rank s and a finite group. Show that $H_2(G, \mathbb{Z})$ can be generated by no more than $r - d + s$ elements.
(b) Show that the braid group on three strings $B_3 = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$ has trivial Schur multiplier. Show that $H_2(\mathbb{Z}^n, \mathbb{Z})$ can be generated by at most $\binom{n}{2}$ elements.

3. By considering the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/|G|\mathbb{Z} \to 0$ show that if G is finite and $H_2(G, \mathbb{Z}) = 0$ then $H_2(G, \mathbb{Z}/|G|\mathbb{Z}) \cong G/G'$. (This suggests that it does not work to compute the Schur multiplier using a finite coefficient module.)

4. (a) Show that the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ is split if and only if $G = 1$.
(b) Show that if G is a free group then $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) \neq 0$.

5. Let G be a finite group. You may assume the following result which was part of question 6 on homework sheet 3: every $\mathbb{Z}G$-module homomorphism $IG \to \mathbb{Z}G$ has image contained in IG.
   (a) Show that every $\mathbb{Z}G$-module homomorphism $IG \to \mathbb{Z}G$ can be factored as $IG \hookrightarrow \mathbb{Z}G \to \mathbb{Z}G$, that is, it can be expressed as the composite of inclusion of IG in $\mathbb{Z}G$ followed by a $\mathbb{Z}G$-module homomorphism $\mathbb{Z}G \to \mathbb{Z}G$. [Apply the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, \mathbb{Z}G)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$.]
   (b) Show that the endomorphism ring $\mathrm{Hom}_{\mathbb{Z}G}(IG, IG)$ is isomorphic to $\mathbb{Z}G/(N)$ where $N = \sum_{g \in G} g$ is the norm element which generates $(N) = (\mathbb{Z}G)^G$. [Use the hint from part (b).]

6. For each of the three crystal structures on the attached sheet determine the point group, and identify the equivalent crystal structure on the list of 17 wallpaper patterns.