Preliminary review sheet for the final exam

The exam is on Friday 12/12/05 from 1:30-4:30 in room Ford Hall 150. I will hold a review session in our usual lecture room from 10:00-12:00 on Friday morning.

There are 12 questions on the exam, some divided into parts, with each question part usually worth 6% of the total. You may not use books or notes. You may use a calculator. Always show your work, and be sure to write down sufficient detail so that I can see that you are able to do all calculations without a calculator if necessary. If you are not sure what is required in any question, or what the question means, do ask.

1. Let $f : \mathrm{Mat}(2,2) \to \mathbb{R}$ be the mapping $f(A) = \mathrm{trace}(A^2)$. Find the directional derivative of f at the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in the direction of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2. Consider the following functions which are defined to be 0 at $(0,0)$. Are they continuous at the origin? Differentiable? Do the partial derivatives exist?

$$\frac{x^2}{\sqrt{x^2+y^2}}, \qquad \frac{2x-5y}{\sqrt{x^2+y^2}}, \qquad \frac{xy}{\sqrt{x^2+y^2}}, \qquad \frac{x^2+y^2}{x+y^2}$$

3. True or false? For each of the following statements, decide whether it is true or false, and then either give brief reasons or a counterexample to justify your assertion.
   (a) There exists a surjective linear mapping $\mathbb{R}^7 \to \mathbb{R}^{10}$.
   (b) If $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function, there can never be a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ with $gf = 1_{\mathbb{R}^2}$, the identity mapping on $\mathbb{R}^2$.
   (c) If $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function, there can never be a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ with $fg = 1_{\mathbb{R}^3}$, the identity mapping on $\mathbb{R}^3$.
   (d) Let $v_1, \ldots, v_r$ be a linearly independent set of vectors in a vector space V and $w_1, \ldots, w_r$ another set of vectors in a vector space W. Then there exists a linear mapping $T : V \to W$ with $T(v_i) = w_i$ for all i with $1 \le i \le r$.
   (e) If S is an $m \times n$ matrix of rank m then there exists an $n \times m$ matrix T with $ST = I$, the identity matrix.
   (f) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and there exists $g : \mathbb{R}^n \to \mathbb{R}^n$ with $fg = gf = 1$. Then g is differentiable.
   (g) If $f : U \to V$ and $g : V \to W$ are linear mappings then $\mathrm{rank}(gf) \le \mathrm{rank}(f)$ always.
   (h) If $S : U \to V$ is a linear mapping which is onto then there exists a linear mapping $T : V \to U$ with $ST = I$.
   (i) If $S : U \to V$ is a linear mapping which is onto then there exists a linear mapping $T : V \to U$ with $TS = I$.
   (j) If $S : U \to V$ is a linear mapping which is 1-1 then there exists a linear mapping $T : V \to U$ with $ST = I$.

(k) If $S : U \to V$ is a linear mapping which is 1-1 then there exists a linear mapping $T : V \to U$ with $TS = I$.

4. Find the number of paths of length 4 from vertex A to itself in the graph

5. Let S be a subset of $\mathbb{R}^n$. We will say that x is a *limit point* of $S \Leftrightarrow$ for all $\epsilon > 0$ there exists $y \in S$ with $0 < |x - y| < \epsilon$. Using the definition that S is closed $\Leftrightarrow$ for every point x not in S there is a ball of some positive radius with center x which contains no point of S, prove that

$$S \text{ is closed} \Leftrightarrow S \supseteq \text{its limit points.}$$

Which of the following statements means x is not a limit point of S?
(i) There exists $\epsilon > 0$ such that for all $y \in S$ either $y = x$ or $|y - x| \geq \epsilon$.
(ii) There exists $\epsilon > 0$ such that for all $y \in S$ either $y = x$ or $|y - x| > \epsilon$.
(iii) There exists $\epsilon > 0$ such that there exists $y \in S$ with either $y = x$ or $|y - x| \geq \epsilon$.
(iv) There exists $y \in S$ such that there exists $\epsilon > 0$ with either $y = x$ or $|y - x| > \epsilon$.

6. Do one step of Newton's method to solve the system of equations

$$\begin{aligned} ye^x + xe^y &= 1 \\ x^3 + xy + \sin y &= 0 \end{aligned} \quad \text{starting at } a_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

7. Calculate $\det \begin{pmatrix} 1 & 2 & -1 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix}$.

8. Prove that

$$Df(a)(h) = \lim_{t \to 0} \frac{f(a + th) - f(a)}{t}.$$