Math 8202 **Homework 1** **PJW**

Date due: February 2, 2009. Hand in only the 5 starred questions.

Section 3.8, page 196 3.86, 3.88, 3.92, 3.93*

A (Review from last semester) (a) Let $p(x) = x^3 + 9x + 6 \in \mathbb{Q}[x]$ and let θ be a root of p in some extension field. Express $(1+\theta)^-1$ as a polynomial in θ.
(b) Same question with $p(x) = x^3 - 2x - 2$.

B Prove directly that the map $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an isomorphism of $\mathbb{Q}(\sqrt{2})$ with itself.

C Determine the degree over $\mathbb{Q}$ of $2 + \sqrt{3}$ and of $1 + \sqrt[3]{2} + \sqrt[3]{4}$.

D* Prove that $\mathbb{Q}(\sqrt{3} + \sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Conclude that $[\mathbb{Q}(\sqrt{3} + \sqrt{5}) : \mathbb{Q}] = 4$. Find an irreducible polynomial satisfied by $\sqrt{3} + \sqrt{5}$, giving justification that your polynomial is irreducible.

E Let F be a field of characteristic $\neq 2$. Let D_1 and D_2 be elements of F, neither of which is a square in F. Prove that $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over F if $D_1 D_2$ is not a square in F and is of degree 2 over F otherwise. When $F(\sqrt{D_1}, \sqrt{D_2})$ is of degree 4 over F the field is called a *biquadratic extension of* F.

F Let F be a field of characteristic $\neq 2$. Let a, b be elements of the field F with b not a square in F. Prove that a necessary and sufficient condition for $\sqrt{a + \sqrt{b}} = \sqrt{m} + \sqrt{n}$ for some m and n in F is that $a^2 - b$ is a square in F. Use this to determine when the field $\mathbb{Q}(\sqrt{a + \sqrt{b}})$ with $a, b \in \mathbb{Q}$ is biquadratic over $\mathbb{Q}$.

G* Let K be an extension of F of degree n.
(a) For any $\alpha \in K$ prove that α acting by left multiplication of K is an F-linear transformation of K.
(b) Prove that K is isomorphic to a subfield of the ring of $n \times n$ matrices of F, so that ring of $n \times n$ matrices over F contains an isomorphic copy of *every* extension of F of degree $\leq n$.

H Let $K = \mathbb{Q}(\sqrt{D})$ for some squarefree integer D. Let $\alpha = a + b\sqrt{D}$ be an element of K. Use the basis $1, \sqrt{D}$ for K as a vector space over $\mathbb{Q}$ and show that the matrix of the linear transformation 'multiplication by α' on K considered in the last exercise has the matrix $\begin{pmatrix} a & bD \\ b & a \end{pmatrix}$. Prove directly that the map $a + b\sqrt{D} \mapsto \begin{pmatrix} a & bD \\ b & a \end{pmatrix}$ is an isomorphism of the field K with a subfield of the ring of 2×2 matrices with coefficients in $\mathbb{Q}$.

I* Determine the splitting field over $\mathbb{Q}$, together with its degree over $\mathbb{Q}$ for each of (a) $x^4 - 2$, (b) $x^4 + 2$, (c) $x^4 + x^2 + 1$ and (d) $x^6 - 4$.

J* (Fall 2000, qn. 5)(12%) Let $K \supseteq k$ be a field extension and $f \in k[X]$ an irreducible polynomial of degree relatively prime to the degree of the field extension $[K : k]$. Show that f is irreducible in $K[X]$.