Math 8202 **Homework 3** **PJW**

Date due: February 16, 2009. Hand in only the 5 starred questions.

Section 4.1, page 217 4.5, 4.8*

T* (Fall 2001, qn. 5) (15%) Let $f(x) = x^4 + ax^3 + bx^2 + ax + 1 \in \mathbb{Q}[x]$, and let E be the splitting field of f.

(a) (3) Show that for each root α of f in E, also α^{-1} is a root of f.

(b) (5) Show that $[E : \mathbb{Q}] \leq 8$.

(c) (7) Show that if f is irreducible in $\mathbb{Q}[x]$ and has exactly two real roots and two complex roots, then the Galois group $G_{E/\mathbb{Q}}$ is isomorphic to D_4, the dihedral group of order 8.

U* (Spring 1995, qn. 3) (15%) Let $f(X) \in k[X]$ be a polynomial of degree n over k and let K be a splitting field for f over k. Suppose that the Galois group $\mathrm{Gal}(K/k)$ is the symmetric group S_n.

(a) (7) Show that f is irreducible in $k[X]$, and separable.

(b) (5) Let α be a root of f in K. Show that in $k(\alpha)[X]$, f factorizes as

$$f(X) = (X - \alpha)g(X)$$

where $g(X)$ is an *irreducible* polynomial.

(c) (3) Determine the Galois group of g over $k(\alpha)$.

V (Spring 2002, qn 5) Let F be a field of characteristic 0. Let $f(x) = x^n - a \in F[x]$, and assume that f does not have a root in F. (But it is *not* assumed that $f(x)$ is irreducible in $F[x]$.) Finally, let E be a splitting field of $f(x)$.

(a) (6%) Show that if F contains a primitive n^{th} root of unity, then the Galois group $G_{E/F}$ is isomorphic to a subgroup of the additive group $\mathbb{Z}/n\mathbb{Z}$.

(b) (6%) Show that if F contains a primitive n^{th} root of unity, then all of the irreducible factors of $f(x)$ in $F[x]$ are of the same degree.

(c) (6%) Show that if F contains a primitive n^{th} root of unity and $g(x)$ is an irreducible factor of $f(x)$ in $F[x]$, then $g(x) = x^k - b$, where k is a divisor of n, b is an element of F, and $b^{n/k} = a$.

(d) (6%) Now let $f(x) = x^6 + a$ in $\mathbb{R}[x]$, where $a > 0$. Determine whether each of the following two statements is true or false:

(i) All of the irreducible factors of $f(x)$ in $\mathbb{R}[x]$ have the same degree.

(ii) If $g(x)$ is an irreducible factor of $f(x)$ in $\mathbb{R}[x]$ then $g(x) = x^k - b$, where k is a divisor of 6 and $b \in \mathbb{R}$ satisfies $b^{6/k} = a$.

W* Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is the splitting field of a separable polynomial of degree 4. Determine the isomorphism type of the Galois group of $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ over $\mathbb{Q}$.

X Show that $\mathbb{Q}(\sqrt{3+\sqrt{5}})$ is the splitting field of a separable polynomial of degree 4. Determine the isomorphism type of the Galois group of $\mathbb{Q}(\sqrt{3+\sqrt{5}})$ over $\mathbb{Q}$.

Y* Determine the Galois group of the splitting field over $\mathbb{Q}$ of $x^8 - 3$.