Math 8202 **Homework 4** **PJW**

Date due: February 23, 2009. There will be a quiz on this date. Hand in only the 5 starred questions.

Section 4.2, page 245 4.17*, 4.18, 4.20

V* (Spring 2002, qn 5) Let F be a field of characteristic 0. Let $f(x) = x^n - a \in F[x]$, and assume that f does not have a root in F. (But it is *not* assumed that $f(x)$ is irreducible in $F[x]$.) Finally, let E be a splitting field of $f(x)$.

(a) (6%) Show that if F contains a primitive n^{th} root of unity, then the Galois group $G_{E/F}$ is isomorphic to a subgroup of the additive group $\mathbb{Z}/n\mathbb{Z}$.

(b) (6%) Show that if F contains a primitive n^{th} root of unity, then all of the irreducible factors of $f(x)$ in $F[x]$ are of the same degree.

(c) (6%) Show that if F contains a primitive n^{th} root of unity and $g(x)$ is an irreducible factor of $f(x)$ in $F[x]$, then $g(x) = x^k - b$, where k is a divisor of n, b is an element of F, and $b^{n/k} = a$.

(d) (6%) Now let $f(x) = x^6 + a$ in $\mathbb{R}[x]$, where $a > 0$. Determine whether each of the following two statements is true or false:

(i) All of the irreducible factors of $f(x)$ in $\mathbb{R}[x]$ have the same degree.

(ii) If $g(x)$ is an irreducible factor of $f(x)$ in $\mathbb{R}[x]$ then $g(x) = x^k - b$, where k is a divisor of 6 and $b \in \mathbb{R}$ satisfies $b^{6/k} = a$.

(Z) Let k be a field.

(a) Show that the mapping $\phi : k[t] \to k[t]$ defined by $\phi(f(t)) = f(at + b)$ for fixed $a, b \in k$, $a \neq 0$ is an automorphism of $k[t]$ which is the identity on k.

(b) Conversely, let ϕ be an automorphism of $k[t]$ which is the identity on k. Prove that there exist $a, b \in k$ with $a \neq 0$ such that $\phi(f(t)) = f(at + b)$ as in (a).

(AAA*) Prove that the automorphisms of the rational function field $k(t)$ which fix k are precisely the *fractional linear transformations* determined by $t \mapsto \frac{at+b}{ct+d}$ for $a, b, c, d \in k$, $ad - bc \neq 0$ (so $f(t) \in k(t)$ maps to $f(\frac{at+b}{ct+d})$).

(BBB*) Determine the fixed field of the automorphism $t \mapsto t + 1$ of $k(t)$.

(CCC*) Let K be an extension of the field F. Let $\phi : K \to K'$ be an isomorphism of K with a field K' which maps F to the subfield F' of K'. Prove that the map $\sigma \mapsto \phi\sigma\phi^{-1}$ defines a group isomorphism $\text{Aut}(K/F) \to \text{Aut}(K'/F')$.