Math 8211 **Homework 3** **PJW**

Date due: Monday October 29, 2012. In class on Wednesday October 31 we will grade your answers, so it is important to be present on that day, with your homework.

Rotman 3.6 (page 114), 5.3(iii) (page 226), 5.6(ii) (page 227), 5.8 (page 227), 5.29 (page 271), 5.31 (page 272).
Questions 1 – 5 below.

1. Let $A = M_{m,m}(R)$ and $B = M_{n,n}(R)$ be matrix rings over R. Show that $A \otimes_R B \cong M_{mn,mn}(R)$, where the ring structure on the tensor product is the one described on page 82.

2. Show that in any commutative diagram of R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

   in which the right hand vertical morphism is the identity and the rows are exact, the left hand square is necessarily a pushout. Also the dual statement.

3. Let F and G be an adjoint pair of functors, so that F is left adjoint to G and G is right adjoint to F. Let η and ϵ be the unit and counit of the adjunction (see page 271). Show that the bijection $\mathrm{Hom}(FA, B) \to \mathrm{Hom}(A, GB)$ equals the mapping given by $f \mapsto G(f) \circ \eta_A$ and its inverse equals $g \mapsto \epsilon_B \circ F(g)$.

4. Let $\mathcal{C}$ be a small category and let $F, G : \mathcal{C} \to \mathrm{Sets}$ be functors. Show that a natural transformation of functors $\tau : F \to G$ is an epimorphism in $\mathrm{Sets}^{\mathcal{C}}$ if and only if for every object x of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a surjection, and it is a monomorphism if and only if for every object x of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a 1-1 map.

5. In question 1 of HW 2 a poset $\mathcal{C}_2$ was constructed from each small category $\mathcal{C}$. Let us write $P(\mathcal{C}) := \mathcal{C}_2$ for the poset so obtained. Recall that $P(\mathcal{C})$ has objects the equivalence classes $\underline{x}$ of objects x of $\mathcal{C}$ under the equivalence relation $x \sim y \Leftrightarrow$ there are morphisms $x \to y$ and $y \to x$. In $P(\mathcal{C})$ there is a unique morphism $\underline{x} \to \underline{y}$ if and only if there is a morphism $x \to y$.
   (a) Show that P may be defined on morphisms of categories (i.e. functors) so as to give a functor P from the category of small categories to the category of posets.
   (b) Consider also the inclusion functor I from the category of posets to the category of small categories. Determine whether I is a left adjoint of P, a right adjoint of P or neither of these.
   (c) On the assumption that there is an adjunction between I and P (in some order), show that the functor $F_2 : \mathcal{C} \to \mathcal{C}_2 = P(\mathcal{C})$ described in HW2 qn. 1 determines

a natural transformation which is either the unit or the count of the adjunction. Determine which of these two it is, and describe the other one. (Recall that the definition of F_2 on objects was $F_2(x) = \underline{x}$).

Extra questions which I was considering, which you should not hand in (because it makes too many questions):

Rotman: 3.18 (page 129), 3.28 (page 151), 3.35 (page 152),

A. Let k be a field, let $\mathcal{D}$ be a small category and let $F : \mathcal{D} \to k$-mod be a diagram of vector spaces over k. Let $\underline{k}$ be the constant functor $\mathcal{D} \to k$-mod which assigns to each object the space k and to each morphism the identity map. Show that $\varprojlim F \leftrightarrow \mathrm{Nat}(\underline{k}, F)$ and $\varinjlim F \leftrightarrow \mathrm{Nat}(F, \underline{k})$

B. Show that a coproduct of projective objects is projective, and a product of injective objects is injective always.

C. Let $\mathcal{C}$ be a small category.
(a) Use Yoneda's lemma to show that representable functors $\mathrm{Hom}_{\mathcal{C}}(x, -)$ are projective objects in $\mathrm{Sets}^{\mathcal{C}}$.
(b) Show that every functor $F : \mathcal{C} \to \mathrm{Sets}$ is the codomain of an epimorphism from a coproduct of representable functors.

Exercises

3.1 Let M be a free R-module, where R is a domain. Prove that if $rm = 0$, where $r \in R$ and $m \in M$, then either $r = 0$ or $m = 0$. (This is false if R is not a domain.)

***3.2** Let R be a ring and let S be a nonzero submodule of a free right R-module F. Prove that if $a \in R$ is not a right zero-divisor[2], then $Sa \neq \{0\}$.

3.3 Define projectivity in **Groups**, and prove that a group G is projective if and only if G is a free group.
Hint. Recall the Nielsen–Schreier Theorem: Every subgroup of a free group is free.

3.4** **(i)** (**Pontrjagin**) If A is a countable torsion-free abelian group each of whose subgroups S of finite rank is free abelian, prove that A is free abelian (the ***rank of an abelian group S is defined as $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} S)$; cf. Exercise 2.36 on page 97).
Hint. See the discussion on page 103.

(ii) Prove that every subgroup of finite rank in $\mathbb{Z}^{\mathbb{N}}$ (the product of countably many copies of $\mathbb{Z}$) is free abelian.

(iii) Prove that every countable subgroup of $\mathbb{Z}^{\mathbb{N}}$ is free. (In Theorem 4.17, we will see that $\mathbb{Z}^{\mathbb{N}}$ itself is not free.)

***3.5** (**Eilenberg**) Prove that every projective left R-module P has a free complement; that is, there exists a free left R-module F such that $P \oplus F$ is free.
Hint. If $P \oplus Q$ is free, consider $Q \oplus P \oplus Q \oplus P \oplus \cdots$.

3.6 Let k be a commutative ring, and let P and Q be projective k-modules. Prove that $P \otimes_k Q$ is a projective k-module.

***3.7** **(i)** Prove that $R = C(\mathbb{R})$, the ring of all real-valued functions on $\mathbb{R}$ under pointwise operations, is not noetherian.

(ii) Recall that $f : \mathbb{R} \to \mathbb{R}$ is a C^∞-function if $\partial^n f / \partial x^n$ exists and is continuous for all n. Prove that $R = C^\infty(\mathbb{R})$, the ring of all C^∞-functions on $\mathbb{R}$ under pointwise operations, is not noetherian.

(iii) If k is a commutative ring, prove that $k[X]$, the polynomial ring in infinitely many indeterminates X, is not noetherian.

***3.8** (**Small**) Let R be the ring of all 2×2 matrices $\left[\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right]$ with $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ is a ring. Schematically, we can describe R as $\left[\begin{smallmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{smallmatrix}\right]$. Prove that R is left noetherian, but that R is not right noetherian.

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[2] An element $a \in R$ is a ***zero-divisor*** if $a \neq 0$ and there exists a nonzero $b \in R$ with $ab = 0$ or $ba = 0$. More precisely, a is a ***right zero-divisor*** if there is a nonzero b with $ba = 0$; that is, multiplication $r \mapsto ra$ is not an injection $R \to R$.

For every open $V \subseteq X$, define $\Gamma(V) = \{\text{continuous } f\colon V \to \mathbb{R}\}$ and, if $V \subseteq W$, where $W \subseteq X$ is another open subset, define $\Gamma(W) \to \Gamma(V)$ to be the restriction map $f \mapsto f|V$. Then properties (i) and (ii) say that $\Gamma(U)$ is the equalizer of the family of maps $\Gamma(U_i) \to \Gamma(U_{ij})$. ◀

Exercises

***5.1** **(i)** Prove that $\varnothing$ is an initial object in **Sets**.

(ii) Prove that any one-point set $\Omega = \{x_0\}$ is a terminal object in **Sets**. In particular, what is the function $\varnothing \to \Omega$?

5.2** A ***zero object in a category $\mathcal{C}$ is an object that is both an initial object and a terminal object.

(i) Prove the uniqueness to isomorphism of initial, terminal, and zero objects, if they exist.

(ii) Prove that $\{0\}$ is a zero object in ${}_R\mathbf{Mod}$ and that $\{1\}$ is a zero object in **Groups**.

(iii) Prove that neither **Sets** nor **Top** has a zero object.

(iv) Prove that if $A = \{a\}$ is a set with one element, then (A, a) is a zero object in $\mathbf{Sets}_*$, the category of pointed sets. If A is given the discrete topology, prove that (A, a) is a zero object in $\mathbf{Top}_*$, the category of pointed topological spaces.

5.3 **(i)** Prove that the zero ring is not an initial object in **ComRings**.

(ii) If k is a commutative ring, prove that k is an initial object in $\mathbf{ComAlg}_k$, the category of all commutative k-algebras.

(iii) In **ComRings**, prove that $\mathbb{Z}$ is an initial object and that the zero ring $\{0\}$ is a terminal object.

5.4 For every commutative ring k, prove that the direct product $R \times S$ is the categorical product in $\mathbf{ComAlg}_k$ (in particular, direct product is the categorical product in $\mathbf{ComAlg}_{\mathbb{Z}} = \mathbf{ComRings}$).

***5.5** Let k be a commutative ring.

(i) Prove that $k[x, y]$ is a free commutative k-algebra with basis $\{x, y\}$.

Hint. If A is any commutative k-algebra, and if $a, b \in A$, there exists a unique k-algebra map $\varphi\colon k[x, y] \to A$ with $\varphi(x) = a$ and $\varphi(y) = b$.

(ii) Use Proposition 5.2 to prove that $k[x] \otimes_k k[y]$ is a free k-algebra with basis $\{x, y\}$.

(iii) Use Proposition 5.4 to prove that $k[x] \otimes_k k[y] \cong k[x, y]$ as k-algebras.

5.6** **(i)** Let Y be a set, and let $\mathcal{P}(Y)$ denote its ***power set; that is, $\mathcal{P}(Y)$ is the partially ordered set of all the subsets of Y. As in Example 1.3(iii), view $\mathcal{P}(Y)$ as a category. If $A, B \in \mathcal{P}(Y)$, prove that the coproduct $A \sqcup B = A \cup B$ and that the product $A \sqcap B = A \cap B$.

(ii) Generalize part (i) as follows. If X is a partially ordered set viewed as a category, and $a, b \in X$, prove that the coproduct $a \sqcup b$ is the least upper bound of a and b, and that the product $a \sqcap b$ is the greatest lower bound.

(iii) Give an example of a category in which there are two objects whose coproduct does not exist.

Hint. Let Ω be a set with at least two elements, and let $\mathcal{C}$ be the category whose objects are its proper subsets, partially ordered by inclusion. If A is a nonempty subset of Ω, then the coproduct of A and its complement does not exist in $\mathcal{C}$.

5.7** Define the ***wedge of pointed spaces $(X, x_0), (Y, y_0) \in \mathbf{Top}_*$ to be $(X \vee Y, z_0)$, where $X \vee Y$ is the quotient space of the disjoint union $X \sqcup Y$ in which the basepoints are identified to z_0. Prove that wedge is the coproduct in $\mathbf{Top}_*$.

5.8 Give an example of a covariant functor that does not preserve coproducts.

***5.9** If A and B are (not necessarily abelian) groups, prove that $A \sqcap B = A \times B$ (direct product) in **Groups**. For readers familiar with group theory, prove that $A \sqcup B = A * B$ (free product) in **Groups**.

***5.10** **(i)** Given a pushout diagram in ${}_R\mathbf{Mod}$:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\beta"] \\ B \arrow[r, "\alpha"'] & D \end{tikzcd}$$

prove that g injective implies α injective and that g surjective implies α surjective. Thus, parallel arrows have the same properties.

(ii) Given a pullback diagram in ${}_R\mathbf{Mod}$:

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

prove that f injective implies α injective and that f surjective implies α surjective. Thus, parallel arrows have the same properties.

Proof. For any integer $n \geq 1$, the free module $P = \bigoplus_{i=1}^{n} R_i$, where $R_i \cong R$, is a small projective generator of $\mathbf{Mod}_R$, and $S = \mathrm{End}_R(P) \cong \mathrm{Mat}_n(R)$. The isomorphism $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_{\mathrm{Mat}_n(R)}$ in Morita's Theorem carries $M \mapsto M \otimes_S P \cong \bigoplus_i M_i$, where $M_i \cong M$ for all i. Hence, if M is a projective right R-module, then $F(M)$ is also projective. But every module in $\mathbf{Mod}_{\mathrm{Mat}_n(R)}$ is projective, by Proposition 4.5 (a ring R is semisimple if and only if every R-module is projective). Therefore, $\mathrm{Mat}_n(R)$ is semisimple, •

There is a lovely part of ring theory, ***Morita theory*** (after K. Morita), developing these ideas. A category $\mathcal{C}$ is isomorphic to a module category if and only if it is an *abelian category* (see Section 5.5) containing a small projective generator P, and which is closed under infinite coproducts (see Mitchell, *Theory of Categories*, p. 104, or Pareigis, *Categories and Functors*, p. 211). Given this hypothesis, then $\mathcal{C} \cong \mathbf{Mod}_S$, where $S = \mathrm{End}(P)$ (the proof is essentially that given for Theorem 5.55). Two rings R and S are called ***Morita equivalent*** if $\mathbf{Mod}_R \cong \mathbf{Mod}_S$. If R and S are Morita equivalent, then $Z(R) \cong Z(S)$; that is, they have isomorphic centers (the proof actually identifies all the possible isomorphisms between the categories). In particular, two *commutative* rings are Morita equivalent if and only if they are isomorphic. See Jacobson, *Basic Algebra* II, pp. 177–184, Lam, *Lectures on Modules and Rings*, Chapters 18 and 19, or Reiner, *Maximal Orders*, Chapter 4.

Exercises

5.29 Give an example of an additive functor $H\colon \mathbf{Ab} \to \mathbf{Ab}$ that has neither a left nor a right adjoint.

***5.30** Let (F, G) be an adjoint pair, where $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$, and let $\tau_{C,D}\colon \mathrm{Hom}(FC, D) \to \mathrm{Hom}(C, GC)$ be the natural bijection.

(i) If $D = FC$, there is a natural bijection

$$\tau_{C,FC}\colon \mathrm{Hom}(FC, FC) \to \mathrm{Hom}(C, GFC)$$

with $\tau(1_{FC}) = \eta_C\colon C \to GFC$. Prove that $\eta\colon 1_{\mathcal{C}} \to GF$ is a natural transformation.

(ii) If $C = GD$, there is a natural bijection

$$\tau^{-1}_{GD,D}\colon \mathrm{Hom}(GD, GD) \to \mathrm{Hom}(FGD, D)$$

with $\tau^{-1}(1_D) = \varepsilon_D\colon FGD \to D$. Prove that $\varepsilon\colon FG \to 1_{\mathcal{D}}$ is a natural transformation. (We call ε the ***counit*** of the adjoint pair.)

5.31 Let (F, G) be an adjoint pair of functors between module categories. Prove that if G is exact, then F preserves projectives; that is, if P is a projective module, then FP is projective. Dually, prove that if F is exact, then G preserves injectives.

5.32 **(i)** Let $F\colon \mathbf{Groups} \to \mathbf{Ab}$ be the functor with $F(G) = G/G'$, where G' is the commutator subgroup of a group G, and let $U\colon \mathbf{Ab} \to \mathbf{Groups}$ be the functor taking every abelian group A into itself (that is, UA regards A as a not necessarily abelian group). Prove that (F, U) is an adjoint pair of functors.

(ii) Prove that the unit of the adjoint pair (F, U) is the natural map $G \to G/G'$.

***5.33** **(i)** If I is a partially ordered set, let $\mathbf{Dir}(I, {}_R\mathbf{Mod})$ denote all direct systems of left R-modules over I. Prove that $\mathbf{Dir}(I, {}_R\mathbf{Mod})$ is a category and that $\varinjlim\colon \mathbf{Dir}(I, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ is a functor.

(ii) In Example 1.19(ii), we saw that constant functors define a functor $|\Box|\colon \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$; to each object C in $\mathcal{C}$ assign the constant functor $|C|$, and to each morphism $\varphi\colon C \to C'$ in $\mathcal{C}$, assign the natural transformation $|\varphi|\colon |C| \to |C'|$ defined by $|\varphi|_D = \varphi$. If $\mathcal{C}$ is cocomplete, prove that $(\varinjlim, |\Box|)$ is an adjoint pair, and conclude that $\varinjlim$ preserves direct limits.

(iii) Let I be a partially ordered set and let $\mathbf{Inv}(I, {}_R\mathbf{Mod})$ denote the class of all inverse systems, together with their morphisms, of left R-modules over I. Prove that $\mathbf{Inv}(I, {}_R\mathbf{Mod})$ is a category and that $\varprojlim\colon \mathbf{Inv}(I, {}_R\mathbf{Mod}) \to {}_R\mathbf{Mod}$ is a functor.

(iv) Prove that if $\mathcal{C}$ is complete, then $(|\Box|, \varprojlim)$ is an adjoint pair and $\varprojlim$ preserves inverse limits.

5.34 **(i)** If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an ascending sequence of submodules of a module A, prove that $A/\bigcup A_i \cong \bigcup A/A_i$; that is, $\operatorname{coker}(\varinjlim A_i \subseteq A) \cong \varinjlim \operatorname{coker}(A_i \to A)$.

(ii) Generalize part (i): prove that any two direct limits (perhaps with distinct index sets) commute.

(iii) Prove that any two inverse limits (perhaps with distinct index sets) commute.

(iv) Give an example in which direct limit and inverse limit do not commute.

5.35 **(i)** Define ACC in ${}_R\mathbf{Mod}$, and prove that if ${}_S\mathbf{Mod} \cong {}_R\mathbf{Mod}$, then ${}_S\mathbf{Mod}$ has ACC. Conclude that if R is left noetherian, then S is left noetherian.