Math 8211 **Homework 6** **PJW**

Date due to me: Thursday December 20, 2012, or earlier. The work could be put in my mail box, or sent to me by email. If you do put work in my mail box, it would help to send me email to alert me to this. If you do this homework assignment I will grade it and it might improve your course grade. It will not lower your course grade below the preliminary grade I assigned you.

From Rotman:

8.2(ii)+ Show that the global dimension of the ring $\mathbb{Z}/n\mathbb{Z}$ is infinite if and only if n is not square-free.

8.3 Let M be a module for a ring R. If $\operatorname{pd} M = n < \infty$ prove that there is a free module F for which $\operatorname{Ext}^n_R(M, F) \neq 0$.

8.5 Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of left R-modules for some ring R. Prove each of the following using the long exact Ext sequence.
  (i) If $\operatorname{pd}(M') < \operatorname{pd}(M)$, prove that $\operatorname{pd}(M'') = \operatorname{pd}(M)$.
  (ii) If $\operatorname{pd}(M') > \operatorname{pd}(M)$, prove that $\operatorname{pd}(M'') = \operatorname{pd}(M') + 1$.
  (iii) If $\operatorname{pd}(M') = \operatorname{pd}(M)$, prove that $\operatorname{pd}(M'') \leq \operatorname{pd}(M') + 1$.

Definitions: In an additive category we say that a mapping $\alpha : U \to V$ is a *kernel* of $\beta : V \to W$ if $\beta\alpha = 0$ and whenever $\gamma : L \to V$ is such that $\beta\gamma = 0$ then there exists a unique mapping $\theta : L \to U$ so that $\gamma = \theta\alpha$. Recall also that a mapping $\alpha : U \to V$ is a *monomorphism* $\Leftrightarrow$ whenever $\alpha f_1 = \alpha f_2$ then $f_1 = f_2$.

Further questions:

1. Let $\alpha : U \to V$ be a morphism in an additive category.
(a) Show that the following two conditions on a morphism are equivalent:
  (i) α has a kernel and it is the morphism $0 \to U$,
  (ii) α is a monomorphism.
(b) Show that if α is the kernel of a morphism β then α is a monomorphism.
[Evidently dual statements hold, but do not write out proofs of these.]
(c) Give an example of a morphism in an additive category which has no kernel.

2. In an exact category, prove that the zero map $0 : X \to Y$ is only an admissible epimorphism if $Y \cong 0$.
Deduce that if a map in an exact category is both an admissible monomorphism and an admissible epimorphism, then it is an isomorphism.

3. Prove that the sequence of complexes $X \xrightarrow{\iota_X} I_X \xrightarrow{\pi_X} X[1]$ defined in the notes is split in each degree. Show also that ι_X is the kernel of π_X. [Do not bother to write out a proof of the dual statement that π_X is the cokernel of ι_X.]

4. Let $C(R\text{-mod})$ be the category whose objects are chain complexes of R-modules, and where the morphisms are chain maps. State and prove a theorem which identifies

completely the projective objects in this category. As an instance of this, give a complete list of all the isomorphism types of indecomposable complexes which are projective in this category in the case where $R = k[X]/(X^3)$ for some field k. [It might help to get started by considering something specific, such as this example.]

5. Let $i : X \to Y$ and $\alpha : X \to Z$ be maps in an exact category, and suppose that i is an admissible monomorphism. Prove that $\binom{i}{\alpha} : X \to Y \oplus Z$ is an admissible monomorphism by writing it as a composite

$$X \xrightarrow{\begin{pmatrix} 1_X \\ \alpha \end{pmatrix}} X \oplus Z \xrightarrow{\begin{pmatrix} i & 0 \\ 0 & 1_Z \end{pmatrix}} Y \oplus Z.$$

6. (a) Show that in an exact category, for every pushout

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{i'} & Y' \end{array}$$

where i is an admissible monomorphism, the cokernels of i and i' may be identified.
(b) Suppose in a Frobenius category we have two admissible short exact sequences $X \to I_1 \to U$ and $X \to I_2 \to V$ where I_1 and I_2 are injective relative to $\mathcal{S}$. Show that $U \cong V$ in the stable category.

7. Write out a proof that in a Frobenius category, given an admissible short exact sequence $X \to Y \to Z$, there is a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \| & & \downarrow & & \downarrow \\ X & \longrightarrow & I_Y \oplus Z & \longrightarrow & X[1] \oplus Z \\ & & \downarrow & & \downarrow \\ & & X[1] & = & X[1] \end{array}$$

in which $X[1]$ is defined so that it appears in an admissible exact sequence $X \to I_Y \to X[1]$, the rows and columns of the diagrams are admissible exact sequences, and the right hand vertical sequence is in fact split. In doing this you may quote any results from the earlier exercises.