Math 8245 **Homework 5** **PJW**

Date due: December 14, 2015. We will go over these questions in class on Dec. 16.

1. By drawing 3 planes cutting a regular cube so that the reflections in these planes generate the group of all isometries of the cube, identify this group of isometries as a Coxeter group. Draw the Coxeter diagram of the group. How big is the group?

2. By considering the effect of the element $s_1 s_2 s_1 s_3$ on the geometric representation show that in the group

$$\langle s_1, s_2, s_3 \mid s_1^2, s_1^2, s_3^2, (s_1 s_2)^3, (s_1 s_3)^3, (s_2 s_3)^3 \rangle$$

this element has infinite order.

3. Let e_i be the ith unit coordinate vector in $\mathbb{R}^n$.
(a) Show that the root system for the Coxeter group W whose diagram has $n-1$ nodes ○——○ ··· ○——○ in a line may be identified with the $n(n-1)$ vectors $e_i - e_j$ with $i \neq j$ in such a way that the simple roots (the vectors α_s) are the $e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}$ in the hyperplane of vectors with coordinate sum zero.
(b) Identify which of the vectors $e_i - e_j$ are positive roots and which are negative roots.
(c) By considering the action of W on the standard $n-1$-simplex which is the convex hull of $e_1, \ldots, e_n$ in $\mathbb{R}^n$, show that $W \cong S_n$.
(d) Letting S_n act on $\{1, \ldots, n\}$ in the usual way, show that if $g \in S_n$ then $\ell(g)$ equals the number of pairs $i < j$ for which $gi > gj$.

4. (Exercise 1 on p. 115 of Humphreys) Given a reduced expression $w = s_1 \cdots s_r$ ($s_i \in S$), set $\alpha_i := \alpha_{s_i}$ and $\beta_i := s_r s_{r-1} \cdots s_{i+1}(\alpha_i)$, interpreting β_r to be α_r. Prove that $\Pi(w)$ (i.e. the set of positive roots sent to negative roots by w) consists of the r distinct positive roots $\beta_1, \ldots, \beta_r$.

5. (Exercise 2 on p. 115 of Humphreys) (a) If W is infinite, prove that the length function takes arbitrarily large values, hence that Φ is infinite. Show that the scalar $-1 \in GL(V)$ does not lie in $\sigma(W)$.
(b) If W is finite, prove that there is one and only one element $w_\circ \in W$ of maximal length, and that $w_\circ$ maps Π onto $-\Pi$.
(c) Let S_n act on $\{1, \ldots, n\}$ in the usual way. Show that

$$w_\circ = (1, n)(2, n-1)(3, n-2) \cdots .$$

6. (Exercise on p. 127 of Humphreys) If the Tits cone U is equal to V^*, prove that W is finite. [Find $w \in W$ for which $w(\bar{C})$ meets $-C$. Then show that $w^{-1}(\alpha_s) < 0$ for all $s \in S$, and deduce that W is finite.]

7. The Coxeter complex of a Coxeter system (W, S) is a 'simplicial complex' whose simplices are the regions $w\overline{C_I}$ where $w \in W$ and I ranges over the subsets of S which are not equal to the whole of S. Here

$$\overline{C_I} = \{f \in V^* \mid \langle f, \alpha_{s_i} \rangle = 0 \text{ for all } i \in I,\ \langle f, \alpha_{s_i} \rangle \geq 0 \text{ for all } i \notin I\}.$$

The subsimplices of $w\overline{C_I}$ are the $w'\overline{C_I}$ where $w'\overline{C_I} \subseteq w\overline{C_I}$.
[This construction really constructs the Coxeter complex in a way similar to an *abstract simplicial complex*. The set of simplices in each dimension is an abstract set, and face relationships are understood between simplices of different dimensions. Here the set of vertices, for instance, is a set of lines in V^*.]
(a) Show that $w'\overline{C_J} \subseteq w\overline{C_I}$ if and only if $J \supset I$ and $w'\overline{C_J} = w\overline{C_J}$. [Assume the theorem in section 5.13 of Humphreys. In particular, note that $\overline{C_I}$ is the fixed point set D^{W_I}.]
(b) Show that the Coxeter complex may also be constructed in the following way: we take the simplices to be in bijection with all cosets wW_I where $w \in W$ and $I \subset S$ is not equal to the whole of S. The subsimplices of wW_I are the cosets $w'W_J$ for which $w'W_J \supseteq wW_I$.
(c) With this description of the Coxeter complex, show that two simplices wW_I and $w'W_J$ meet in a (non-empty) Simlex if and only if there is a third coset $w''W_K$ which contains them both. Show that in this case $I \cup J \subseteq K$ and $w''W_K = wW_K = w'W_K$. Identify which cosets are the vertices of the Coxeter complex, and also which cosets are the simplices of maximal dimension.
(d) Assume without proof that the maximal dimension of a simplex in the Coxeter complex is $|S| - 1$, and that every simplex is contained in a simplex of this dimension. How many simplices of dimension $|S| - 1$ contain each simplex of dimension $|S| - 2$?