1. (2 pts) Let $M$ be a $kG$-module. Show that $M$ admits a non-singular $G$-invariant bilinear form if and only if $M \cong M^*$ as $kG$-modules.

Solution: Suppose $\langle -, - \rangle$ is a non-singular $G$-invariant bilinear form on $M$. Then $\theta : M \to M^*$ given by $\theta(x)(v) = \langle x, v \rangle$ is an isomorphism of vector spaces by something done in class. Furthermore $\theta(xg)(v) = \langle xg, v \rangle = \langle x, vg^{-1} \rangle = \theta(x)(vg^{-1}) = (\theta(x))^g(V)$ so $\theta(xg) = (\theta(x))^g$ for all $x \in M$ and $g \in G$. This means $\theta$ is a homomorphism of $kG$-modules.

Conversely if $M \cong M^*$ as $kG$-modules, let $\psi : M \to M^*$ be a $kG$-module isomorphism. Then $\langle x, v \rangle := \psi(x)(v)$ is a bilinear form, and by something done in class it is non-singular. The same algebra as above shows that it is $G$-invariant.

2. Let $M$ be a $kG$-module and let $\mathcal{B}$ be the vector space of bilinear forms $M \times M \to k$.
   a) (2 pts) For each $g \in G$ we may construct two new bilinear forms $\langle -, - \rangle_1^g : v, w \mapsto \langle vg, wg \rangle$, and $\langle -, - \rangle_2^g : v, w \mapsto \langle vg^{-1}, wg^{-1} \rangle$. One of these definitions makes $\mathcal{B}$ into a $kG$-module via $\langle -, - \rangle \cdot g = \langle -, - \rangle_i^g$, $i = 1$ or $2$. Which value of $i$ achieves this?

Solution: We show that $\langle -, - \rangle_2^{gh} = (\langle -, - \rangle_2^g)_2^h$.
The left sends $v, w \mapsto \langle vh^{-1}g^{-1}, wh^{-1}g^{-1} \rangle$.
The right sends $v, w \mapsto \langle vh^{-1}, wh^{-1} \rangle_2^g = \langle vh^{-1}g^{-1}, wh^{-1}g^{-1} \rangle$ and these are equal. Thus $\langle -, - \rangle_2^g$ does the trick.

   c) (2 pts) Taking a standard basis for $M$ and for $\mathcal{B}$ we may express a bilinear form $f$ by its Gram matrix $A_f$, and the action of $g \in G$ on $M$ by its matrix $\rho(g)$. Which of the following gives the right action of $G$ on $\mathcal{B}$ (pun intended): (i) $A_f \mapsto \rho(g)^T A_f \rho(g)$, or (ii) $A_f \mapsto \rho(g) A_f \rho(g)^T$?

Solution: At this point I realize I am in big trouble with right actions and left actions. If I have group elements acting from the right then they must act on row vectors, and in order to have linear maps act on these also they must also act from the right, unless we do something crazy with transposes. Let's say vectors are row vectors.

Neither possibility seems to be correct. Since

$$\langle v, w \rangle_2^g = \langle v\rho(g^{-1}), w\rho(g^{-1}) \rangle_2^g = v\rho(g^{-1}) A_f \rho(g^{-1})^T w^T$$

we should have $A_f \mapsto \rho(g^{-1}) A_f \rho(g^{-1})^T$.

3. Let $G = C_3 = \langle g \rangle$ be cyclic of order 3 and let $k = \mathbb{F}_3$. We define $M_2 = ke_1 \oplus ke_2$ to be a 2-dimensional space acted on by $g$ via the matrix $\rho(g) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
   a) (1 pts) Find the matrix via which $g$ acts on the space $\mathcal{B}$ of bilinear forms $M \times M \to k$.

Solution: With the convention of 2c $g$ acts via $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} A_f \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and computing the effect of this on a basis for bilinear forms gives a matrix $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) (2 pts) Show that the space of $G$-invariant bilinear forms has dimension 2.

Solution: We compute the nullspace of the last matrix minus the identity, i.e.

$$\begin{bmatrix} 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and it has dimension 2, with basis $[0, 0, 0, 1]$ and $[0, 1, -1, 0]$ corresponding to bilinear forms $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

c) (1 pts) Show that $M_2 \cong M_2^*$ as $kG$-modules and find a $G$-invariant non-degenerate form on $M_2$.

Solution: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is a $G$-invariant non-singular bilinear form, so by question 1 $M \cong M^*$.

d) (2 pts) Show that $M_2$ does not admit any *symmetric* $G$-invariant non-degenerate bilinear form, but that it does admit a skew-symmetric such form.

Solution: The form just shown is skew-symmetric. The general $G$-invariant form is $\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix}$ and this is symmetric if and only if $a = 0$, but then the form is singular.

4. (1 pts) Let $U$ be a $kG$-submodule of the $kG$-module $M$. Show that $U^\circ$ is a $kG$-submodule of $M^*$.

Solution: $U^\circ = \{f \mid (u)f = 0 \text{ for all } u \in U\}$. Now $f \in U^\circ$ implies $uf^g = (ug^{-1})fg = 0$ since $ug^{-1} \in U$. Thus $f^g \in U^\circ$ for all $g \in G$ and $U^\circ$ is a $kG$-submodule.

(3 pts) Suppose further that $M$ comes supplied with a non-singular $G$-invariant bilinear form. Show that $U^\perp \cong U^\circ$ as $kG$-modules. Deduce that the isomorphism type of $U^\perp$ is independent of the choice of non-singular $G$-invariant bilinear form.

Solution: The map $\theta : M \to M^*$of question 1 is a map of $kG$-modules, and we have seen that $\theta(x) \in U^\circ$ if and only if $x \in U^\perp$. Thus $\theta$ restricts to an isomorphism of $kG$-modules $U^\perp \to U^\circ$. Since $U^\circ$ is independent of the choice of bilinear form, so is the isomorphism type of $U^\perp$.

5. (2 pts) Let $H$ be a subgroup of a group $G$, and write

$$H\backslash G = \{Hg \mid g \in G\}$$

for the set of right cosets of $H$ in $G$. There is a permutation action of $G$ on this set from the right, namely $(Hg_1)g_2 = Hg_1g_2$. Let $\overline{H} = \sum_{h \in H} h \in kG$ denote the sum of the elements of $H$, as an element of the group ring of $G$. Show that the permutation module $k[H\backslash G]$ is isomorphic as an $kG$-module to the submodule $\overline{H}\cdot kG$ of $kG$.

Solution: We define $k[H\backslash G] \to \overline{H}\cdot kG$ by $Hg \mapsto \overline{H}g$ on the basis elements $Hg$ of $k[H\backslash G]$. This is well defined since $Hg = Hg_1$ if and only if $g_1 = hg$ for some $h \in H$, so $\overline{H}g_1 = \overline{h}g = \overline{H}g$. It is surjective since $\overline{H}\cdot kG$ is spanned by the elements $\overline{H}g$, $g \in G$. It is one to one since the distinct elements $\overline{H}g$ are obtained by letting $g$ range over a set of coset representatives, and since these elements have disjoint supports in $G$, they form a basis for $\overline{H}\cdot kG$ in bijection with the basis of $k[G\backslash g]$.

6. (3=1+2 pts) Let $V$ be the subspace of the 10-dimensional space $k^{10}$ over the field $k$ which has as a basis the vectors

$$\begin{array}{llllllllll}
[0, & 1, & -1, & -1, & 1, & 0, & 0, & 0, & 0, & 0] \\
[1, & 0, & -1, & -1, & 0, & 1, & 0, & 0, & 0, & 0] \\
[0, & 1, & -1, & 0, & 0, & 0, & -1, & 1, & 0, & 0] \\
[1, & 0, & -1, & 0, & 0, & 0, & -1, & 0, & 1, & 0] \\
[1, & 0, & 0, & 0, & -1, & 0, & -1, & 0, & 0, & 1].
\end{array}$$

With respect to this basis of $V$, write down the Gram matrix for the bilinear form on $V$ which is the restriction of the standard bilinear form on $k^{10}$. Supposing further that $k$ has characteristic 3, determine the dimension of the space $V/(V \cap V^{\perp})$.

Solution: It is

$$\begin{bmatrix} 4 & 2 & 2 & 1 & -1 \\ 2 & 4 & 1 & 2 & 1 \\ 2 & 1 & 4 & 2 & 1 \\ 1 & 2 & 2 & 4 & 2 \\ -1 & 1 & 1 & 2 & 4 \end{bmatrix}$$

Over $\mathbb{F}_3$ this is

$$\begin{bmatrix} 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

This has rank 1, so $\dim V/(V \cap V^{\perp}) = 1$.