The diagonal entries of B^3 correspond to the fact that there are exactly 10 loops of length 3 going from any given vertex back to itself.

1.2.20

1.2.21 a. The proof is the same as with unoriented walks (proposition 1.2.23): first we state that if B_n is the $n \times n$ matrix whose i, jth entry is the number of walks of length n from V_i to V_j, then $B_1 = A^1 = A$ for the same reasons as in the proof of proposition 1.2.23. Here again if we assume the proposition true for n, we have:

$$(B_{n+1})_{i,j} = \sum_{k=1}^{n} (B_n)_{i,k}(B_1)_{k,j} = \sum_{k=1}^{n} (A^n)_{i,k} A_{k,j} = (A^{n+1})_{i,j}.$$

So $A^n = B_n$ for all n.

b. If the adjacency matrix is upper triangular, then you can only go from a lower number vertex to a higher number vertex; if it is lower triangular, you can only go from a higher number vertex to a lower number vertex. If it is diagonal, you can never go from any vertex to any other.

Solution 1.2.21, part b: The first column of an adjacency corresponds to vertex 1, the second to vertex 2, and so on, and the same for the rows. If the matrix is upper triangle, for example, $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, you can go from vertex 1 to vertex 2, but not from 2 to 1; from 2 to 3, but not from 3 to 2, and so on: once you have gone from a lower-numbered vertex to a higher-numbered vertex, there is no returning.

1.2.22

1.2.23 a. Let $A = \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AB = I$.

b. Whatever matrix one multiplies B by on the right, the top left corner of the resultant matrix will always be 0 when we need it to be 1. So the matrix B has no right inverse.

c. With A and B as in part a, write $I^\top = I = AB = (AB)^\top = B^\top A^\top$. So $A^\top$ is a right inverse for $B^\top$, so $B^\top$ has infinitely many right inverses.

1.3.1 a. Every linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ is given by a 2×4 matrix. For example, $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 7 \end{bmatrix}$ is a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$.

b. Any row matrix 3 wide will do, for example, $[1, -1, 2]$; such a matrix takes a vector in $\mathbb{R}^3$ and gives a number.

1.3.2 a. 3×2 (3 high, 2 wide); b. 3×3; c. 2×4; d. 1×4 (a row matrix with four entries)

1.3.3 a. $\mathbb{R}^4 \to \mathbb{R}^3$ b. $\mathbb{R}^2 \to \mathbb{R}^5$ c. $\mathbb{R}^4 \to \mathbb{R}^2$ d. $\mathbb{R}^4 \to \mathbb{R}$.

1.3.4 a. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

1.3.5 Multiply the original matrix by the vector $\vec{\mathbf{v}} = \begin{bmatrix} .25 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .025 \\ .5 \end{bmatrix}$, putting $\vec{\mathbf{v}}$ on the right.

1.3.6 The only one characterized by linearity is (b).

1.3.7 It is enough to know what T gives when evaluated on the three standard basis vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The matrix of T is $\begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

1.3.8 a. Five questions: what are $T\vec{\mathbf{e}}_1$, $T\vec{\mathbf{e}}_2$, $T\vec{\mathbf{e}}_3$, $T\vec{\mathbf{e}}_4, T\vec{\mathbf{e}}_5$, where the $\vec{\mathbf{e}}_i$ are the five standard basis vectors in $\mathbb{R}^5$? The matrix is

$$[T\vec{\mathbf{e}}_1, T\vec{\mathbf{e}}_2, T\vec{\mathbf{e}}_3, T\vec{\mathbf{e}}_4, T\vec{\mathbf{e}}_5].$$

b. Six questions: what does T give when evaluated on the six standard basis vectors in $\mathbb{R}^6$?

c. No. For example, you could evaluate T on $2\vec{\mathbf{e}}_i$, for the appropriate $\vec{\mathbf{e}}_i$, and divide the answer by 2, to get the ith column of the matrix.

1.3.9 No, T is not linear. If it were, the matrix would be $[T] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, but $[T] \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 5 \end{bmatrix}$, which contradicts the definition of the transformation.

1.3.10 Yes there is; its matrix is $\begin{bmatrix} 3 & 1 & -2 \\ 0 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}$. Since by the definition of linearity $T(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T(\vec{\mathbf{v}}) + T(\vec{\mathbf{w}})$, we have

$$T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

1.3.11 The rotation matrix is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. This transformation takes $\vec{\mathbf{e}}_1$ to $\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$, which is thus the first column of the matrix, by Theorem 1.3.4; it takes $\vec{\mathbf{e}}_2$ to $\begin{bmatrix} \cos(90^\circ-\theta)=\sin\theta \\ \sin(90^\circ-\theta)=\cos\theta \end{bmatrix}$, which is the second column.

One could also write this transformation as the rotation matrix of Example 1.3.9, applied to $-\theta$:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

1.3.12 a. The matrices corresponding to S and T are

$$[S]=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [T]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The matrix of the composition $S\circ T$ is given by matrix multiplication:

$$[S\circ T]=[S][T]=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

b. The matrices $[S\circ T]$ and $[T\circ S]$ are inverses of each other: you can either compute it out, or compose:

$$T\circ(S\circ S)\circ T = T\circ T = I \quad \text{and} \quad S\circ(T\circ T)\circ S = S\circ S = I.$$

Since S and T are reflections, $S\circ S = T\circ T = I$.

1.3.13 The expressions a, e, f, and j are not well-defined compositions. For the others:

b. $C\circ B:\mathbb{R}^m\to\mathbb{R}^n$ (domain $\mathbb{R}^m$, codomain $\mathbb{R}^n$) c. $A\circ C:\mathbb{R}^k\to\mathbb{R}^m$

d. $B\circ A\circ C:\mathbb{R}^k\to\mathbb{R}^k$ g. $B\circ A:\mathbb{R}^n\to\mathbb{R}^k$

h. $A\circ C\circ B:\mathbb{R}^m\to\mathbb{R}^m$ i. $C\circ B\circ A:\mathbb{R}^n\to\mathbb{R}^n$

1.3.14 The transformation is given by $T\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x+1 \\ y+1 \end{pmatrix}$. It is an affine translation but not linear because $T\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; a linear transformation must take the origin to the origin. To see why this requirement is necessary, consider $T(\mathbf{x})=T(\mathbf{x}+\mathbf{0})=T(\mathbf{x})+T(\mathbf{0})$, so $T(\mathbf{0})$ must be 0. For instance, in this case,

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix}+T\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}+\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \text{but} \quad T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}+\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)=T\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

1.3.15 We need to show that $A(\vec{\mathbf{v}}+\vec{\mathbf{w}}) = A\vec{\mathbf{v}}+A\vec{\mathbf{w}}$ and that $A(c\vec{\mathbf{v}}) = cA\vec{\mathbf{v}}$. By definition 1.2.4,

$$(A\vec{\mathbf{v}})_i = \sum_{k=1}^{n} a_{i,k}v_k, \quad (A\vec{\mathbf{w}})_i = \sum_{k=1}^{n} a_{i,k}w_k, \text{ and}$$

$$\begin{aligned}\big(A(\vec{\mathbf{v}}+\vec{\mathbf{w}})\big)_i &= \sum_{k=1}^{n} a_{i,k}(v+w)_k = \sum_{k=1}^{n} a_{i,k}(v_k+w_k) \\ &= \sum_{k=1}^{n} a_{i,k}v_k + \sum_{k=1}^{n} a_{i,k}w_k = (A\vec{\mathbf{v}})_i + (A\vec{\mathbf{w}})_i.\end{aligned}$$

Solution 1.3.15: With the dot product, introduced in the next section, the solution is simpler: Every entry in the vector $A\vec{\mathbf{v}}$ is the dot product of $\vec{\mathbf{v}}$ with one of the rows of A. The dot product is linear with respect to both of its arguments, so the mapping $A\vec{\mathbf{v}}$ is also linear with respect to $\vec{\mathbf{v}}$.

Similarly, $\big(A(c\vec{\mathbf{v}})\big)_i = \sum_{k=1}^{n} a_{i,k}(cv)_k = \sum_{k=1}^{n} a_{i,k}cv_k = c\sum_{k=1}^{n} a_{i,k}v_k = c(A\vec{\mathbf{v}})_i.$

1.3.16

$$\begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix} = \begin{bmatrix}\cos(\theta_2) & -\sin(\theta_2)\\ \sin(\theta_2) & \cos(\theta_2)\end{bmatrix}\begin{bmatrix}\cos(\theta_1) & -\sin(\theta_1)\\ \sin(\theta_1) & \cos(\theta_1)\end{bmatrix}$$

So:

$$\begin{aligned}&\begin{bmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{bmatrix}\\ &= \begin{bmatrix}\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\cos(\theta_2)-\sin(\theta_2)\cos(\theta_1)\\ \sin(\theta_1)\cos(\theta_2)+\sin(\theta_2)\cos(\theta_1) & \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\end{bmatrix}.\end{aligned}$$

Thus by identification we deduce:

$$\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)$$

$$\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2)+\sin(\theta_2)\cos(\theta_1).$$

1.3.17

$$\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{bmatrix}^2 = \begin{bmatrix}\cos(2\theta)^2+\sin(2\theta)^2 & \cos(2\theta)\sin(2\theta)-\sin(2\theta)\cos(2\theta)\\ \sin(2\theta)\cos(2\theta)-\cos(2\theta)\sin(2\theta) & \sin(2\theta)^2+\cos(2\theta)^2\end{bmatrix}$$

$$\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{bmatrix}^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$$

1.3.18

1.3.19 By commutativity of matrix addition: $\frac{AB+BA}{2} = \frac{BA+AB}{2}$ so the Jordan product is commutative. By non-commutativity of matrix multiplication:

$$\begin{aligned}\frac{\frac{AB+BA}{2}C + C\frac{AB+BA}{2}}{2} &= \frac{ABC+BAC+CAB+CBA}{4}\\ &\neq \frac{ABC+ACB+BCA+CBA}{4} = \frac{A\frac{BC+CB}{2} + \frac{BC+CB}{2}A}{2}\end{aligned}$$

so the Jordan product is not associative.

1.3.20 a. $\mathrm{Re}\,(tz_1 + uz_2) = \mathrm{Re}\,(ta_1 + ua_2 + i(tb_1 + ub_2) = ta_1 + ua_2 = t\mathrm{Re}\,(z_1) + u\mathrm{Re}\,(z_2)$ $(t, u \in \mathbb{R})$.

b. $\mathrm{Im}\,(tz_1 + uz_2) = \mathrm{Im}\,(ta_1 + ua_2 + i(tb_1 + ub_2) = tb_1 + ub_2 = t\,\mathrm{Im}\,(z_1) + u\,\mathrm{Im}\,(z_2)$ $(t, u \in \mathbb{R})$.

c. $c(tz_1 + uz_2) = c(ta_1 + ua_2 + i(tb_1 + ub_2) = ta_1 + ua_2 - i(tb_1 + ub_2) = t(a_1 - ib_1) + u(a_2 - ib_2) = tc(z_1) + uc(z_2)$ $(t, u \in \mathbb{R})$.

d. $m_w(tz_1 + uz_2) = w(tz_1 + uz_2) = w \times tz_1 + w \times uz_2 = tm_w(z_1) + tm_w(z_2)$ $(t, u \in \mathbb{R})$.

1.3.21 The number 0 is in the set, since $\mathrm{Re}\,(0) = 0$. If a, b are in the set, then $a+b$ is also in the set, since $\mathrm{Re}\,(a+b) = \mathrm{Re}\,(a) + \mathrm{Re}\,(b) = 0$. If a is in the set and c is a real number, then ca is in the set, since $\mathrm{Re}\,(ca) = c\mathrm{Re}\,(a) = 0$. So the set is a subspace of $\mathbb{C}$. The subspace is a line in $\mathbb{C}$ with a polar angle θ such that $\theta + \varphi = k\pi/2$, where φ is the polar angle of w and k is an odd integer.

1.4.1 a. Numbers: $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$, $|\vec{\mathbf{v}}|$, $|A|$, and $\det A$. (If A consists of a single row, then $A\vec{\mathbf{v}}$ is also a number.)

Vectors: $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$ and $A\vec{\mathbf{v}}$ (unless A consists of a single row).

So far we have defined only determinants of 2×2 and 3×3 matrices; in section 4.8 we will define the determinant in general.

b. In the expression $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$, the vectors must each have three entries. In the expression $\det A$, the matrix A must be square.

1.4.2

a. $\left|\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right| = \sqrt{1^2 + 2^2} = \sqrt{5}$. b. $\left|\begin{bmatrix} \sqrt{2} \\ \sqrt{7} \end{bmatrix}\right| = \sqrt{2 + 7} = 3$.

c. $\left|\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right| = \sqrt{1 + 1 + 1} = \sqrt{3}$. d. $\left|\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right| = \sqrt{1 + 4 + 4} = 3$.

1.4.3 To normalize a vector, divide it by its length. This gives:

a. $\dfrac{1}{\sqrt{17}}\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ b. $\dfrac{1}{\sqrt{58}}\begin{bmatrix} -3 \\ 7 \end{bmatrix}$ c. $\dfrac{1}{\sqrt{31}}\begin{bmatrix} \sqrt{2} \\ -2 \\ -5 \end{bmatrix}$

1.4.4 a. Let α be the required angle. Then

$$\cos\alpha = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} \\ \sqrt{7} \end{bmatrix}}{\left|\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right|\left|\begin{bmatrix} \sqrt{2} \\ \sqrt{7} \end{bmatrix}\right|} = \frac{\sqrt{2} + 2\sqrt{7}}{3\sqrt{5}} \approx .9996291\ldots,$$

and $\alpha \approx \arccos .9996291 \cdots \approx .027235\ldots$ radians $\approx 1.5606°$.

So those two vectors are remarkably close to collinear.

b. Let β be the required angle. Then

$$\cos\beta = \frac{\begin{bmatrix}1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\2\end{bmatrix}}{\left|\begin{bmatrix}1\\-1\\1\end{bmatrix}\right|\left|\begin{bmatrix}1\\-2\\2\end{bmatrix}\right|} = \frac{5}{3\sqrt{3}} \approx .962250\ldots,$$

and $\beta = \arccos .962250\cdots \approx .27564\ldots \text{rad} \approx 15.793°$.

1.4.5 a. $\cos(\theta) = \frac{1}{1\times\sqrt{3}}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \frac{1}{\sqrt{3}}$, so

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right) \approx .95532.$$

b. $\cos(\theta) = 0$, so $\theta = \pi/2$.

1.4.6

a. $\det\begin{bmatrix}3 & 1\\0 & 2\end{bmatrix} = (3\times 2) - (1\times 0) = 6$ b.(b) $\det\begin{bmatrix}1 & 0 & 2\\2 & 4 & 1\\0 & 1 & 3\end{bmatrix} = 15$

(c) $\det\begin{bmatrix}-2 & 5 & 3\\-1 & 3 & 4\\-2 & 3 & 7\end{bmatrix} = -14$ d. $\det\begin{bmatrix}1 & 2 & -6\\0 & 1 & -3\\1 & 0 & -2\end{bmatrix} = -2-6+6 = -2$

1.4.7 a. det $= 1$; the inverse is $\begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}$

b. det $= 0$; no inverse

c. det $= ad$; if $a, d \neq 0$, the inverse is $\dfrac{1}{ad}\begin{bmatrix}d & -b\\0 & a\end{bmatrix}$.

d. det $= 0$; no inverse

1.4.8 a. det $= -4$ b. det $= adf$ c. det $= g(ad-bc)$

1.4.9 a. $\begin{bmatrix}-6yz\\3xz\\5xy\end{bmatrix}$ b. $\begin{bmatrix}6\\7\\-4\end{bmatrix}$ c. $\begin{bmatrix}-2\\-22\\3\end{bmatrix}$

1.4.10 a. By proposition 1.4.11,

$$|A^k| \le |A||A^{k-1}| \le |A|^2|A^{k-2}| \le \cdots \le |A|^k.$$

For $A = \begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}$, we have $|A^3| = \sqrt{223} \approx 15$ and $|A|^3 = 7\sqrt{7} \approx 18.55$.

b. The first statement is true. Since $\vec{\mathbf{v}} = -2\vec{\mathbf{u}}$, by theorem 1.4.5,

$$|\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}| = |\vec{\mathbf{u}}^\top\vec{\mathbf{v}}| = |\vec{\mathbf{u}}^\top||\vec{\mathbf{v}}| = |\vec{\mathbf{u}}||\vec{\mathbf{v}}|.$$