

Date due: October 24, 2018.

1. Use GAP to show that

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ca)^5 = 1 \rangle \cong A_5 \times C_2.$$

2. The generalized quaternion group of order 2^n has a presentation

$$\langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1} \rangle.$$

Use GAP to investigate the generalized quaternion group of order 32. Get a list of the orders of the elements. Compute the derived subgroup and the center. Draw a picture of the lattice of subgroups of this group. What is the minimum degree of a faithful permutation representation of this group?

3. Write out a proof of this theorem: Let G be a group for which there is a transitive finite G -set Ω with the following property: for every G -set Ψ there is a homomorphism of G -sets $\Omega \rightarrow \Psi$. Then G is finite, and $|G| = |\Omega|$.
4. (Question 11.4 from Rotman's book) A group G is *injective* \Leftrightarrow whenever we are given a subgroup A of a group B and a homomorphism $f : A \rightarrow G$ there exists a homomorphism $g : B \rightarrow G$ so that the restriction of g to A is f . Prove that injective groups have order 1. [Hint (D.L. Johnson): let A be free on $\{a, b\}$ and let $B = A \rtimes \langle c \rangle$ where c has order 2 and $cac^{-1} = b, cbc^{-1} = a$.]
5. (Question 11.5 from Rotman's book) Let $X = \{x_k \mid k \in K\}$ and let $Y \subseteq X$. If F is free on X and H is the normal subgroup generated by Y , show that F/H is free.
6. (Question 11.6 from Rotman's book) Show that a free group F on $\{x, y\}$ has an automorphism f with $f(f(a)) = a$ for all $a \in F$ and with the further property that $f(a) = a$ if and only if $a = 1$.

Extra Questions: do not hand in

7. Use GAP to show that $SL(2, 5)$ has a normal subgroup of order 2 such that the quotient is isomorphic to A_5 . Show that $SL(2, 5)$ has no subgroup isomorphic to A_5 . Identify the Sylow 2-subgroups of $SL(2, 5)$.

8. Use GAP to investigate the groups

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ac)^4 = 1 \rangle$$

and

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ac)^3 = 1 \rangle$$

In each case identify the quotient by the center $G/Z(G)$ and determine whether or not $G = Z(G) \times H$ for some subgroup H .

9. Let F be a free group of rank 2. Show that it is possible to find a set of three elements that generate F , no two of which generate F .