Math 8246 **Homework 1** **PJW**

Date due: Wednesday February 6, 2019

1. Show that the two extensions $0 \to \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/3\mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \xrightarrow{\mu'} \mathbb{Z} \xrightarrow{\epsilon'} \mathbb{Z}/3\mathbb{Z} \to 0$ are not equivalent, where $\mu = \mu'$ is multiplication by 3, $\epsilon(1) \equiv 1 \pmod{3}$ and $\epsilon'(1) \equiv 2 \pmod{3}$.

2. (D&F 10.4, 4) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]

3. (D&F 10.4, 5) Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

4. (D&F 10.4, 6) If R is any integral domain with quotient field Q, prove that

$$(Q/R) \otimes_R (Q/R) = 0.$$

5. (D&F 10.4, 11) Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

6. Show that, as a ring, $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ is the direct sum of two fields. [The ring multiplication is $(a \otimes b)(c \otimes d) := ac \otimes bd$ on basic tensors. See Proposition 19 of D&F. Assume question 25 from 10.4 of D&F.]

7. (D&F 10.5, 14(a)) Let $0 \to L \xrightarrow{\psi} M \xrightarrow{\phi} N \to 0$ be a sequence of R-modules.
   (a) Prove that the associated sequence

$$0 \to \mathrm{Hom}_R(D, L) \xrightarrow{\psi'} \mathrm{Hom}_R(D, M) \xrightarrow{\phi'} \mathrm{Hom}_R(D, N) \to 0$$

   is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence. [Assume that Hom is left exact: do not prove this. To show the sequence splits, take $D = N$ and show the lift of the identity map in $\mathrm{Hom}_R(N, N)$ to $\mathrm{Hom}_R(N, M)$ is a splitting homomorphism for ϕ.]
   (b) Do not bother with this part of the question. It is a similar statement obtained by applying $\mathrm{Hom}_R(-, D)$ to the short exact sequence.