Math 8246 **Homework 4 (version 2)** **PJW**
Date due: Wednesday May 1, 2019.

1. For each of the crystal structures labeled 2 and 3 on the sheet with three Escher designs (lizards and frogs, angels and bats), determine the point group, and identify the equivalent crystal structure on the list of 17 wallpaper patterns.

2. Provide a proof that there are two arithmetic crystal classes of wallpaper patterns with point group D_6. Show that the class in which D_6 acts via the matrices
$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$
gives rise to the crystal structure p31m (rather than p3m1).

3. EITHER: Give an explanation of the flow chart hand-out for determining which wallpaper pattern you are looking at. Your explanation should do the following: 1. Reconcile the two notations for wallpaper patterns in use, so that you can label the flow chart with outputs such as pm, pg, p2mm, p2mg etc. 2. Describe what things like '3 Centers' mean. 3. Explain what $m = 4$ etc mean (that is easier!).
OR: Create your own flow chart, with your own criteria that distinguish the 17 patterns, adequately explained.
[You may wish to consider things like a fundamental domain for the action of T and how lines of reflection and points of rotation are distributed vis-à-vis the fundamental domain.]

4. Making sure that the generators you use for the action on $\mathbb{Z}^2$ have the order you think they have, choose a presentation for D_8 and compute $H^2(D_8, T)$, where T is the only lattice $\mathbb{Z}^2$ on which D_8 acts. Identify which of the two space groups p4mm and p4gm correspond to the split extension, and compute the largest possible point stabilizer in the non-split extension, in its action on 2-dimensional space.

5. Let R be a (not necessarily commutative) ring. Show that every homomorphism of free left R-modules $\phi : R^p \to R^q$ determines, and is determined by, a matrix $A_\phi = (a_{i,j})$ with the property that if $\mathbf{v} = (v_1, \ldots, v_p)^T$ is a column vector $\mathbf{v} \in R^p$ then $\phi(\mathbf{v}) = A\mathbf{v}$, where $A\mathbf{v}$ has entries $\sum_j v_j a_{i,j}$. Show that if $B_\psi = (b_{k,i})$ is the matrix of another map $\psi : R^q \to R^r$ then the composite $\psi\phi$ has matrix with entries $(\sum_i a_{i,j} b_{k,i})$.

Extra: Investigate KaleidoPaint at http://www.geometrygames.org/ and the image gallery http://www.geometrygames.org/KaleidoPaint/Gallery/index.html

6. Don't do this question: Show that the group with the Coxeter presentation
$$\langle s_1, s_2, s_3 \mid s_1^2, s_1^2, s_3^2, (s_1 s_2)^3, (s_1 s_3)^3, (s_2 s_3)^3 \rangle$$
is a crystallographic group in dimension 2. Identify it on the list of 17 wallpaper patterns. Find generators for the translation subgroup as words in the Coxeter generators s_1, s_2, s_3.