Math 8300 **Homework 2** **PJW**

Date due: Wednesday October 9, 2019

Assume all categories considered are small. A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ so that GF is naturally isomorphic to the identity functor $1_{\mathcal{C}}$ and FG is naturally isomorphic to $1_{\mathcal{D}}$ (meaning that there are invertible natural transformations $\tau : GF \to 1_{\mathcal{C}}$ and $\sigma : FG \to 1_{\mathcal{D}}$).

1. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of small categories.
   (a) Show that, for all objects $x, y \in \mathrm{Ob}\mathcal{C}$, F provides a bijection
   $$\mathrm{Hom}_{\mathcal{C}}(x, y) \leftrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x), F(y)),$$
   so that $\mathrm{End}_{\mathcal{C}}(x) \cong \mathrm{End}_{\mathcal{D}}(F(x))$ as monoids.
   (b) Show that $x \cong y$ in $\mathcal{C}$ if and only if $F(x) \cong F(y)$ in $\mathcal{D}$.
   (c) Let $\mathcal{E}$ be a further category. Show that the functor categories $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ are naturally equivalent.
2. Let $\mathcal{C}$ be a category and let $x, y \in \mathrm{Ob}\mathcal{C}$. Prove that if $x \cong y$ then $\mathrm{Hom}_{\mathcal{C}}(x, -)$ and $\mathrm{Hom}_{\mathcal{C}}(y, -)$ are naturally isomorphic functors.
3. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors and $\eta : F \to G$ a natural transformation.
   (a) Show that if, for all $x \in \mathrm{Ob}\mathcal{C}$, the mapping $\eta_x : F(x) \to G(x)$ is an isomorphism in $\mathcal{D}$, then η is a natural isomorphism (meaning that it has a 2-sided inverse natural transformation $\theta : G \to F$).
   (b) Suppose that F is an equivalence of categories and that F is naturally isomorphic to G, so $F \simeq G$. Show that G is an equivalence of categories.
4. Let $\mathcal{C}$ be a small category. A *self-equivalence* of $\mathcal{C}$ is an equivalence of categories $F : \mathcal{C} \to \mathcal{C}$. Show that the set of natural isomorphism classes of self equivalences of $\mathcal{C}$ is a group, with multiplication induced by composition of functors.
5. Let G be a group, which we regard as a category $\mathcal{G}$ with a single object, and with the elements of G as morphisms. Let $F : \mathcal{G} \to \mathcal{G}$ be a functor.
   (a) Show that F is naturally isomorphic to the identity functor $1_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ if and only if the mapping $F : G \to G$, induced by F on the set of morphisms, is an inner automorphism; that is, an automorphism of the form $c_g : G \to G$ for some $g \in G$, where $c_g(h) = ghg^{-1}$ for all $h \in G$.

(b) Show that self equivalences of $\mathcal{G}$ are automorphisms of $\mathcal{G}$.

(c) Optional: do not hand in. Show that the group of natural isomorphism classes of self equivalences of $\mathcal{G}$ is isomorphic to $\operatorname{Aut}(G)/\operatorname{Inn}(G)$. (In the context of group theory, $\operatorname{Inn}(G)$ denotes the set of inner automorphisms of G, and $\operatorname{Aut}(G)/\operatorname{Inn}(G)$ is called the group of *outer* (or *non-inner*) automorphisms.)

6. Let I be the poset with two elements 0 and 1, and with $0 < 1$. If P and Q are posets we can regard them as categories $\mathcal{P}$ and $\mathcal{Q}$ whose objects are the elements of the posets, and where there is a unique morphism $x \to y$ if and only if $x \le y$.

   (a) Show that if P and Q are posets then a functor $\mathcal{P} \to \mathcal{Q}$ is 'the same thing as' an order-preserving map.

   (b) Now consider two order-preserving maps $f, g : P \to Q$, and regard them as functors $F, G : \mathcal{P} \to \mathcal{Q}$. Show that the following three conditions are equivalent:

      i. there is a natural transformation $\tau : F \to G$,

      ii. $f(x) \le g(x)$ for all $x \in P$,

      iii. there is an order-preserving map $h : P \times I \to Q$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in \mathcal{P}$. Here $P \times I$ denotes the product poset with order relation $(a_1, b_1) \le (a_2, b_2)$ if and only if $a_1 \le a_2$ and $b_1 \le b_2$, where $a_i \in P$ and $b_i \in I$.

7. Let K be a field. Show that the category Vec of finite dimensional vectors spaces over K is equivalent to the category $\mathcal{C}$ with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$, where $\operatorname{Hom}_{\mathcal{C}}(n, m)$ is the set $M_{m,n}(K)$ of $m \times n$ matrices with entries in K, and where composition of morphisms is matrix multiplication. In case m or n is zero, give a definition of $\operatorname{Hom}_{\mathcal{C}}(n, m)$ that will make this question make sense.