bound by $\alpha\beta = 0$. Then $\Gamma(\operatorname{mod} A)$ is given by

$$\begin{tikzcd}[column sep=small, row sep=small]
 & {}_1{}^1{}_0 \arrow[dr] & & {}_0{}^1{}_1 \arrow[dr] & \\
{}_1{}^0{}_0 \arrow[ur] \arrow[dr] & & {}_1{}^2{}_1 \arrow[ur] \arrow[dr] & & {}_0{}^0{}_1 \\
 & {}_1{}^{/1}{}_1 \arrow[ur] \arrow[dr] & & {}_1{}^{1\backslash}{}_1 \arrow[ur] \arrow[dr] & \\
{}_0{}^1{}_0 \arrow[ur] & & {}_1{}^0{}_1 \arrow[ur] & & {}_0{}^1{}_0
\end{tikzcd}$$

where modules are replaced by their dimension vectors and one must identify the two copies of $S(2) = {}_0{}^1{}_0$, thus forming a cycle. Here, ${}_1{}^{/1}{}_1$ represents the indecomposable projective module $P(3) = \begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl, "0"'] & \\ K & & K \arrow[ul, "1"'] \arrow[ll, "1"] \end{tikzcd}$, while ${}_1{}^{1\backslash}{}_1$ represents the indecomposable injective module $I(1) = \begin{tikzcd}[column sep=tiny, row sep=tiny] & K \arrow[dl, "1"'] & \\ K & & K \arrow[ul, "0"'] \arrow[ll, "1"] \end{tikzcd}$. It follows that indecomposable modules are not uniquely determined by their dimension vectors, because $P(3) \ncong I(1)$ and $\operatorname{\mathbf{dim}} P(3) = \operatorname{\mathbf{dim}} I(1)$.

IV.5. The first Brauer–Thrall conjecture

At the origin of many recent developments of representation theory are the following two conjectures attributed to Brauer and Thrall.

Conjecture 1. *A finite dimensional K-algebra is either representation–finite or there exist indecomposable modules with arbitrarily large dimension.*

Conjecture 2. *A finite dimensional algebra over an infinite field K is either representation–finite or there exists an infinite sequence of numbers $d_i \in \mathbb{N}$ such that, for each i, there exists an infinite number of nonisomorphic indecomposable modules with K-dimension d_i.*

The first statement has now been shown to hold true, whenever the field K is arbitrary (see [13], [14], [140], [147], [148], [151], [154], [170]), and the second one when K is algebraically closed (see [26], [27], [124], [140], [162], and for historical notes see [83]). Our objective in this section is to give a simple proof of the first conjecture.

Let A be a finite dimensional K-algebra. A sequence of irreducible morphisms in $\operatorname{mod} A$ of the form

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t$$

with all the M_i indecomposables is called a **chain of irreducible morphisms** from M_0 to M_t of length t.

5.1. Lemma. *Let $t \in \mathbb{N}$ and let M and N be indecomposable right A-modules with $\operatorname{Hom}_A(M, N) \neq 0$. Assume that there exists no chain of irreducible morphisms from M to N of length $< t$.*

(a) *There exists a chain of irreducible morphisms*

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_t} M_t$$

and a homomorphism $g : M_t \to N$ with $gf_t \dots f_2 f_1 \neq 0$.

(b) *There exists a chain of irreducible morphisms*

$$N_t \xrightarrow{g_t} N_{t-1} \xrightarrow{g_{t-1}} \cdots \longrightarrow N_1 \xrightarrow{g_1} N_0 = N$$

and a homomorphism $f : M \to N_t$ with $g_1 \dots g_t f \neq 0$.

Proof. We only prove (a); the proof of (b) is similar. We proceed by induction on t. For $t = 0$, there is nothing to show. Assume thus that M and N are given with $\operatorname{Hom}_A(M, N) \neq 0$ and that there is no chain of irreducible morphisms from M to N of length $< t + 1$. By the induction hypothesis, there exists a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t$$

and a homomorphism $g : M_t \to N$ with $gf_t \dots f_1 \neq 0$. The induction hypothesis implies that g cannot be an isomorphism. Because M_t and N are indecomposable, g is not a section. We consider the left minimal almost split morphism starting with M_t

$$h = \begin{bmatrix} h_1 \\ \vdots \\ h_1 \end{bmatrix} : M_t \longrightarrow \bigoplus_{j=1}^{s} L_j,$$

where the modules $L_1, \dots, L_s$ are indecomposable. Then g factors through h, that is, there exists $u = [u_1, \dots, u_s] : \bigoplus_{j=1}^{s} L_j \longrightarrow N$ such that $g = uh = \sum_{j=1}^{s} u_j h_j$. Thus, because $0 \neq gf_t \dots f_1 = \sum_{j=1}^{s} u_j h_j f_t \dots f_1$, there exists j such that $1 \leq j \leq s$ and $u_j h_j f_t \dots f_1 \neq 0$. Setting $M_{t+1} = L_j$, $f_{t+1} = h_j$ and $g' = u_j$, our claim follows from the fact that h_j is irreducible. $\square$

5.2. Lemma (Harada and Sai). *For a natural number b, let*

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \rightarrow \cdots \rightarrow M_{2^b-1} \xrightarrow{f_{2^b-1}} M_{2^b}$$

be a chain of nonzero nonisomorphisms in $\operatorname{mod} A$*, with all* M_i *indecomposables of length* $\leq b$*. Then* $f_{2^b-1}\dots f_2 f_1 = 0$.

Proof. We show by induction on n that if

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \ \to \ \cdots \ \to \ M_{2^n-1} \xrightarrow{f_{2^n-1}} M_{2^n}$$

is a sequence of nonzero nonisomorphisms between indecomposable modules of length $\leq b$, then the length of the image of the composite homomorphism $f_{2^n-1}\dots f_2 f_1$ is $\leq b-n$. This will imply the statement upon setting $b=n$.

Let $n=1$. If the length $\ell(\operatorname{Im} f_1)$ of $\operatorname{Im} f_1$ is equal to b, then f_1 is an isomorphism, a contradiction that shows that $\ell(\operatorname{Im} f_1) \leq b-1$. Assume that the statement holds for n, and let

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \to M_{2^n-1} \xrightarrow{f_{2^n-1}} M_{2^n} \xrightarrow{f_{2^n}} M_{2^n+1} \xrightarrow{f_{2^n+1}} \cdots \xrightarrow{f_{2^{n+1}-1}} M_{2^{n+1}}$$

be a sequence of nonzero nonisomorphisms between indecomposable modules of length $\leq b$. We consider the two homomorphisms $f = f_{2^n-1}\dots f_2 f_1$ and $h = f_{2^{n+1}-1}\dots f_{2^n+1}$. By the induction hypothesis, $\ell(\operatorname{Im} f) \leq b-n$ and $\ell(\operatorname{Im} h) \leq b-n$. If at least one of these two inequalities is strict, we are done. We may thus suppose that $\ell(\operatorname{Im} f) = \ell(\operatorname{Im} h) = b-n > 0$. Let $g = f_{2^n}$. We must show that $\ell(\operatorname{Im} hgf) \leq b-n-1$.

We claim that if this is not the case, then g is an isomorphism, a contradiction that completes the proof. Assume thus that $\ell(\operatorname{Im} hgf) > b-n-1$. Because $\ell(\operatorname{Im} hgf) \leq \ell(\operatorname{Im} f) = b-n$, this implies that $\ell(\operatorname{Im} hgf) = b-n$. Now

$$\ell(\operatorname{Im} hgf) = \ell\left(\frac{\operatorname{Im} f}{\operatorname{Im} f \cap \operatorname{Ker} hg}\right) = \ell(\operatorname{Im} f) - \ell(\operatorname{Im} f \cap \operatorname{Ker} hg).$$

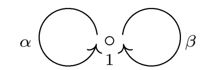
This implies that $\ell(\operatorname{Im} f \cap \operatorname{Ker} hg) = 0$, hence $\operatorname{Im} f \cap \operatorname{Ker} hg = 0$. On the other hand, $\operatorname{Im} hgf \subseteq \operatorname{Im} hg \subseteq \operatorname{Im} h$ and $\ell(\operatorname{Im} hgf) = \ell(\operatorname{Im} h) = b-n$ give $\ell(\operatorname{Im} hg) = b-n$. Consequently,

$$\ell(\operatorname{Ker} hg) = \ell(M_{2^n}) - \ell(\operatorname{Im} hg) = \ell(M_{2^n}) - (b-n) = \ell(M_{2^n}) - \ell(\operatorname{Im} f).$$

This shows that $M_{2^n} = \operatorname{Im} f \oplus \operatorname{Ker} hg$. Because M_{2^n} is indecomposable and $f \neq 0$, we have $\operatorname{Ker} hg = 0$. Therefore hg is a monomorphism. Hence g itself is a monomorphism. Similarly, one shows that $\operatorname{Im} gf \cap \operatorname{Ker} h = 0$, hence that $M_{2^n+1} = \operatorname{Im} gf \oplus \operatorname{Ker} h$. Because $gf \neq 0$ and the module M_{2^n+1} is indecomposable then we get $M_{2^n+1} = \operatorname{Im} gf$, so that gf and therefore g are epimorphisms. This completes the proof that g is an isomorphism, and hence of the lemma. □

The following example shows that the bounds given in the Harada–Sai lemma are the best bounds possible.

5.3. Example. Let A be given by the quiver



consisting of two loops α and β, bound by $\alpha^2 = 0$, $\beta^2 = 0$, $\alpha\beta = 0$, and $\beta\alpha = 0$.

We construct 7 indecomposable A-modules of length ≤ 3 and 6 nonisomorphisms between them with nonzero composition.

The algebra A admits a unique simple module S_A and any A-module can be written in a form of a triple $(V, \varphi_\alpha, \varphi_\beta)$, where V is a finite dimensional K-vector space and $\varphi_\alpha, \varphi_\beta : V \to V$ are K-linear endomorphisms satisfying the conditions $\varphi_\alpha^2 = 0$, $\varphi_\beta^2 = 0$, $\varphi_\alpha\varphi_\beta = \varphi_\beta\varphi_\alpha = 0$, and a morphism $(V, \varphi_\alpha, \varphi_\beta) \to (V', \varphi'_\alpha, \varphi'_\beta)$ is a K-linear map $f : V \to V'$ such that $\varphi'_\alpha f = f\varphi_\alpha$ and $\varphi'_\beta f = f\varphi_\beta$. Let thus

$$\begin{array}{ccccccc}
M_1 & = & M_5 & = & A_A & = & \left(K^3, \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\1&0&0 \end{bmatrix}\right), \\
M_2 & = & M_6 & = & A_A/S & = & \left(K^2, \begin{bmatrix} 0&0\\1&0 \end{bmatrix}, 0\right), \\
M_3 & = & M_7 & = & (DA)_A & = & \left(K^3, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}\right), \\
M_4 & = & S_A & = & (K,0,0). & &
\end{array}$$

Each of these modules has a simple top or a simple socle and hence is indecomposable. Let now

$$\begin{array}{lllllll}
f_1 = \begin{bmatrix} 1&0&0\\0&0&1 \end{bmatrix} : M_1 \longrightarrow M_2, & & f_2 = \begin{bmatrix} 0&1\\0&0\\1&0 \end{bmatrix} : M_2 \longrightarrow M_3, \\
f_3 = \begin{bmatrix} 1&0&0 \end{bmatrix} : M_3 \longrightarrow M_4, & & f_4 = \begin{bmatrix} 0\\0\\1 \end{bmatrix} : M_4 \longrightarrow M_5, \\
f_5 = \begin{bmatrix} 1&0&0\\0&0&1 \end{bmatrix} : M_5 \longrightarrow M_6, & & f_6 = \begin{bmatrix} 1&0\\0&0\\0&1 \end{bmatrix} : M_6 \longrightarrow M_7.
\end{array}$$

It is easily checked that each of these matrices defines an A-module homomorphism, and $f_6 f_5 f_4 f_3 f_2 f_1 = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix} \neq 0$.

We are now able to prove our criterion of representation–finiteness, which was announced in the previous section and implicitly used in the construction of Auslander–Reiten quivers.

5.4. Theorem. *Assume that* A *is a basic and connected finite dimensional* K*-algebra. If* $\Gamma(\operatorname{mod} A)$ *admits a connected component* $\mathcal{C}$ *whose* *modules are of bounded length, then $\mathcal{C}$ is finite and $\mathcal{C} = \Gamma(\operatorname{mod} A)$. In particular, A is representation–finite.*

Proof. Let b be a bound for the length of the indecomposable modules X with $[X]$ in $\mathcal{C}$. Let M, N be two indecomposable A-modules such that $\operatorname{Hom}_A(M, N) \neq 0$. If $[M] \in \mathcal{C}_0$, there exists a chain of irreducible morphisms from M to N of length smaller than $2^b - 1 = t$, and in particular $[N] \in \mathcal{C}_0$. Indeed, if this is not the case, there exists, by (5.1), a chain of irreducible morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \cdots \to M_{t-1} \xrightarrow{f_t} M_t$$

and a homomorphism $g : M_t \to N$ with $g f_t \dots f_1 \neq 0$. However, (5.2) yields $f_t \dots f_1 = 0$, a contradiction that shows our claim. Similarly, if $[N] \in \mathcal{C}_0$, we have $[M] \in \mathcal{C}_0$.

Let now $[M] \in \mathcal{C}_0$ be arbitrary. There exists an indecomposable projective module P_A such that $\operatorname{Hom}_A(P, M) \neq 0$; hence we also have $[P] \in \mathcal{C}_0$. It follows from (II.3.4) and (I.5.17) that, for any other indecomposable projective P', there exists a sequence of indecomposable projective modules $P = P_0, P_1, \dots, P_s = P'$ such that $\operatorname{Hom}_A(P_{i-1}, P_i) \neq 0$ or $\operatorname{Hom}_A(P_i, P_{i-1}) \neq 0$ for each $1 \leq i \leq s$, because the algebra A is connected, $P \cong e_a A$ and $P' \cong e_b A$ for some primitive orthogonal idempotents e_a, e_b of A, and (I.4.2) yields $\operatorname{Hom}_A(e_a A, e_b A) \cong e_b A e_a$. Hence $[P'] \in \mathcal{C}_0$. We deduce that any indecomposable A-module X corresponds to a point $[X]$ in $\mathcal{C}$, because there exists an indecomposable projective A-module P' such that $\operatorname{Hom}_A(P', X) \neq 0$. This shows that $\mathcal{C} = \Gamma(\operatorname{mod} A)$.

On the other hand, for each indecomposable projective A-module P and each indecomposable A-module M such that $\operatorname{Hom}_A(P, M) \neq 0$, we know that there exists a chain of irreducible morphisms from P to M of length smaller than $t = 2^b - 1$. Because there are only finitely many nonisomorphic indecomposable projectives, there are only finitely many nonisomorphic indecomposable modules corresponding to points in $\mathcal{C}$. Hence A is representation–finite. □

As a consequence of (5.4) we get the validity of the first Brauer–Thrall conjecture.

5.5. Corollary. *Any algebra is either representation–finite or admits indecomposable modules of arbitrary length.* □

We end this section with the following corollary, which underlines the importance of the irreducible morphisms and hence of the Auslander–Reiten quiver, for the description of the module category of a representation–finite algebra.

5.6. Corollary. *Let A be a representation–finite algebra. Any nonzero nonisomorphism between indecomposable modules in* $\operatorname{mod} A$ *is a sum of compositions of irreducible morphisms.*

Proof. Let M, N be indecomposable A-modules and $t \geq 1$. Denote by $\operatorname{rad}_A^t(M, N)$ the K-subspace of $\operatorname{rad}_A(M, N)$ consisting of the K-linear combinations of compositions $f_1 f_2 \dots f_t$, where $f_1, f_2, \dots, f_t$ are nonisomorphisms between indecomposable A-modules. Because A is representation–finite, the lengths of the indecomposable A-modules are bounded; hence, by the Harada–Sai lemma (5.2), there exists $m \geq 1$ such that $\operatorname{rad}_A^{m+1}(M, N) = 0$ for all M and N.

Let $g \in \operatorname{rad}_A(M, N)$ be nonzero. If $g \notin \operatorname{rad}_A^2(M, N)$, then g is irreducible and there is nothing to prove. If $g \in \operatorname{rad}_A^2(M, N)$, there exists s such that $2 \leq s \leq m$ and $g \in \operatorname{rad}_A^s(M, N) \setminus \operatorname{rad}_A^{s+1}(M, N)$.

We prove our statement by descending induction on s. If $s = m$, then g is a sum of nonzero compositions $g_1 \cdot g_2 \cdot \ldots \cdot g_m$ of nonisomorphisms $g_1, g_2, \dots, g_m$ between indecomposable modules. Because $\operatorname{rad}_A^{m+1}(M, N) = 0$, the homomorphisms $g_1, \dots, g_m$ do not belong to the square of the radical and therefore are irreducible. This proves the statement for $s = m$. Suppose that $s \leq m - 1$. Then g is a sum of nonzero compositions $g_1 g_2 \dots g_s$ of nonisomorphisms between indecomposable modules. Let g' denote the sum of all the summands $g_1 g_2 \dots g_s$ of g in which all the homomorphisms $g_1, g_2, \dots, g_s$ are irreducible. Then $g'' = g - g' \in \operatorname{rad}_A^{s+1}(M, N)$. If $g'' = 0$, the statement is trivial. If $g'' \neq 0$, then, by the induction hypothesis, g'' is a sum of compositions of irreducible morphisms and therefore so is $g = g' + g''$. The proof is now complete. □

IV.6. Functorial approach to almost split sequences

Let A be a finite dimensional K-algebra. We present in this section an interpretation of the almost split sequences in $\operatorname{mod} A$ in terms of the projective resolutions of the simple objects in the categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$ of the contravariant, and covariant, respectively, K-linear functors from the category $\operatorname{mod} A$ of finitely generated right A-modules into the category $\operatorname{mod} K$ of finite dimensional K-vector spaces. These categories are defined in Section A.2 of the Appendix and are both seen to be abelian. We recall that, given a pair of functors F and G in the category $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$), we denote by $\operatorname{Hom}(F, G)$ the set of functorial morphisms

$\varphi : F \to G$.

Of particular interest in our study is the following classical result.

6.1. Theorem (Yoneda's lemma). *Let $\mathcal{C}$ be an additive K-category and X be an object in $\mathcal{C}$.*

(a) *For any contravariant functor $F : \mathcal{C} \longrightarrow \operatorname{mod} K$, the correspondence $\pi : \varphi \mapsto \varphi_X(1_X)$ defines a bijection between the set $\operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F)$ of functorial morphisms $\varphi : \operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$ and the set $F(X)$.*

(b) *For any covariant functor $F : \mathcal{C} \longrightarrow \operatorname{mod} K$, the correspondence $\pi : \varphi \mapsto \varphi_X(1_X)$ defines a bijection between the set $\operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(X,-), F)$ of functorial morphisms $\varphi : \operatorname{Hom}_{\mathcal{C}}(X,-) \longrightarrow F$ and the set $F(X)$.*

Proof. We only prove (a); the proof of (b) is similar. For a functorial morphism $\varphi : \operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$, we have $\varphi_X(1_X) \in F(X)$, so π defines a map $\operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F) \longrightarrow F(X)$. We now construct its inverse

$$\sigma : F(X) \longrightarrow \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F).$$

Let $a \in F(X)$ and Y be an arbitrary object in $\mathcal{C}$. We define the map $\sigma(a)_Y : \operatorname{Hom}_{\mathcal{C}}(Y,X) \longrightarrow F(Y)$ to be given by $\sigma(a)_Y(f) = F(f)(a)$, for $f \in \operatorname{Hom}_{\mathcal{C}}(Y,X)$.

To show that $\sigma(a) : \operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$ is a functorial morphism, we must show that, for any morphism $g : Y \to Z$, the following diagram is commutative

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(Y,X) \arrow[r, "\sigma(a)_Y"] & F(Y) \\
\operatorname{Hom}_{\mathcal{C}}(Z,X) \arrow[r, "\sigma(a)_Z"] \arrow[u, "\operatorname{Hom}_{\mathcal{C}}(g,X)"] & F(Z) \arrow[u, "F(g)"']
\end{tikzcd}$$

Let thus $f \in \operatorname{Hom}_{\mathcal{C}}(Z,X)$; then $F(g)\sigma(a)_Z(f) = F(g)F(f)(a) = F(f \circ g)(a)$, while $\sigma(a)_Y \operatorname{Hom}_{\mathcal{C}}(g,X)(f) = \sigma(a)_Y(f \circ g) = F(f \circ g)(a)$.

It remains to show that π and σ are mutually inverse.

(i) Let $a \in F(X)$. To prove that $\pi\sigma(a) = a$, we note that

$$\pi\sigma(a) = \sigma(a)_X(1_X) = F(1_X)(a) = 1_{F(X)}(a) = a.$$

(ii) Let $\varphi \in \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F)$. To prove that $\sigma\pi(\varphi) = \varphi$, we show that, for any object Y in $\mathcal{C}$, we have $\sigma\pi(\varphi)_Y = \varphi_Y$. By definition, for any $f \in \operatorname{Hom}_{\mathcal{C}}(Y,X)$, we have

$$\sigma\pi(\varphi)_Y(f) = F(f)(\pi(\varphi)) = F(f)\varphi_X(1_X).$$

Because φ is a functorial morphism, the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(X,X) \arrow[r, "\varphi_X"] \arrow[d, "\operatorname{Hom}_{\mathcal{C}}(f,X)"'] & F(X) \arrow[d, "F(f)"] \\
\operatorname{Hom}_{\mathcal{C}}(Y,X) \arrow[r, "\varphi_Y"] & F(Y)
\end{tikzcd}$$

That is, $F(f)\varphi_X = \varphi_Y \operatorname{Hom}_{\mathcal{C}}(f,X)$. Thus we have

$$\sigma\pi(\varphi)_Y(f) = \varphi_Y \operatorname{Hom}_{\mathcal{C}}(f,X)(1_X) = \varphi_Y(f)$$

and the proof is complete. $\square$

6.2. Corollary. *Let $\mathcal{C}$ be an additive K-category and let X be an object in $\mathcal{C}$.*

(a) *Let F be a subfunctor of $\operatorname{Hom}_{\mathcal{C}}(-,X)$. The map $f \mapsto \operatorname{Hom}_{\mathcal{C}}(-,f)$ is a bijection $F(X) \cong \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), F)$. In particular, for any object Y in $\mathcal{C}$, the map $\operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(-,X), \operatorname{Hom}_{\mathcal{C}}(-,Y))$ given by $f \mapsto \operatorname{Hom}_{\mathcal{C}}(-,f)$ is a bijection.*

(b) *Let F be a subfunctor of $\operatorname{Hom}_{\mathcal{C}}(X,-)$. The map $f \mapsto \operatorname{Hom}_{\mathcal{C}}(f,-)$ is a bijection $F(X) \cong \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(X,-), F)$. In particular, for any object Y in $\mathcal{C}$, the map $\operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}(\operatorname{Hom}_{\mathcal{C}}(Y,-), \operatorname{Hom}_{\mathcal{C}}(X,-))$ given by $f \mapsto \operatorname{Hom}_{\mathcal{C}}(f,-)$ is a bijection.*

Proof. We only prove (a); the proof of (b) is similar. Let $f \in F(X) \subseteq \operatorname{Hom}_{\mathcal{C}}(X,X)$. It was shown that the inverse of the bijection π in Yoneda's lemma 6.1 is given by $\sigma(f) : \operatorname{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$. We show that $\sigma(f) = \operatorname{Hom}_{\mathcal{C}}(-,f)$. Indeed, let Y be an object in $\mathcal{C}$ and $g \in \operatorname{Hom}_{\mathcal{C}}(Y,X)$; then $\sigma(f)_Y(g) = F(g)(f) = f \circ g = \operatorname{Hom}_{\mathcal{C}}(Y,f)(g)$ because, by definition, $F(g) \in F(Y) \subseteq \operatorname{Hom}_{\mathcal{C}}(Y,X)$. This shows the first assertion. The second follows from the first applied to the functor $F = \operatorname{Hom}_{\mathcal{C}}(-,Y)$. $\square$

In particular, it follows from (6.2) that the categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$ are not only abelian, they are also additive K-categories. As a second corollary, we now show that a Hom functor uniquely determines the representing object.

6.3. Corollary. *Let $\mathcal{C}$ be an additive K-category and let X, Y be two objects in $\mathcal{C}$.*

(a) *$X \cong Y$ if and only if $\operatorname{Hom}_{\mathcal{C}}(-,X) \cong \operatorname{Hom}_{\mathcal{C}}(-,Y)$.*

(b) *$X \cong Y$ if and only if $\operatorname{Hom}_{\mathcal{C}}(X,-) \cong \operatorname{Hom}_{\mathcal{C}}(Y,-)$.*

Proof. We only prove (a); the proof of (b) is similar. Clearly, $X \cong Y$ implies $\operatorname{Hom}_{\mathcal{C}}(-,X) \cong \operatorname{Hom}_{\mathcal{C}}(-,Y)$. Conversely, assume that there is an isomorphism $\operatorname{Hom}_{\mathcal{C}}(-,X) \cong \operatorname{Hom}_{\mathcal{C}}(-,Y)$ of functors. By (6.2), there exist morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(-,f) : \operatorname{Hom}_{\mathcal{C}}(-,X) \to \operatorname{Hom}_{\mathcal{C}}(-,Y)$ and $\operatorname{Hom}_{\mathcal{C}}(-,g) : \operatorname{Hom}_{\mathcal{C}}(-,Y) \to \operatorname{Hom}_{\mathcal{C}}(-,X)$

are mutually inverse functorial isomorphisms. Thus the equalities $\mathrm{Hom}_{\mathcal{C}}(-,1_X) = 1_{\mathrm{Hom}_{\mathcal{C}}(-,X)} = \mathrm{Hom}_{\mathcal{C}}(-,g)\circ\mathrm{Hom}_{\mathcal{C}}(-,f) = \mathrm{Hom}_{\mathcal{C}}(-,g\circ f)$ give $g\circ f = 1_X$, by (6.2) again. Similarly, $f\circ g = 1_Y$. □

An object P in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$) is said to be **projective** if for any functorial epimorphism $\varphi : F \to G$, the induced map of K-vector spaces $\mathrm{Hom}(P,\varphi) : \mathrm{Hom}(P,F) \longrightarrow \mathrm{Hom}(P,G)$, given by $\psi \mapsto \varphi\psi$, is surjective.

We now observe that Yoneda's lemma also gives projective objects in the categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$.

6.4. Corollary. *Let A be a K-algebra and M be a module in* $\mathrm{mod}\, A$.

(a) *The functor* $\mathrm{Hom}_A(-,M)$ *is a projective object in* $\mathcal{F}un^{op} A$.

(b) *The functor* $\mathrm{Hom}_A(M,-)$ *is a projective object in* $\mathcal{F}un A$.

Proof. We only prove (a); the proof of (b) is similar. We must prove that, for any functorial epimorphism $\varphi : F \to G$, the induced map

$$\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi) : \mathrm{Hom}(\mathrm{Hom}_A(-,M),F) \longrightarrow \mathrm{Hom}(\mathrm{Hom}_A(-,M),G)$$

given by $\psi \mapsto \varphi\psi$, is surjective. We claim that the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathrm{Hom}_A(-,M),F) \arrow[r, "\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi)"] \arrow[d, "\pi^F"', "\cong"] & \mathrm{Hom}(\mathrm{Hom}_A(-,M),G) \arrow[d, "\pi^G", "\cong"'] \\
F(M) \arrow[r, "\varphi_M"] & G(M)
\end{tikzcd}$$

is commutative, where π^F and π^G denote the bijection π in Yoneda's lemma 6.1 applied to F and G, respectively. Indeed, let $\psi \in \mathrm{Hom}(\mathrm{Hom}_A(-,M),F)$, then

$$\begin{aligned}
\varphi_M \pi^F(\psi) &= \varphi_M\psi_M(1_M) = (\varphi\psi)_M(1_M) = \pi^G(\varphi\psi) \\
&= \pi^G\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi)(\psi).
\end{aligned}$$

On the other hand, φ_M is surjective, because φ is a functorial epimorphism. Hence so is $\mathrm{Hom}(\mathrm{Hom}_A(-,M),\varphi)$. □

A functor F in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$) is called **finitely generated** if F is isomorphic to a quotient of a functor of the form $\mathrm{Hom}_A(-,M)$ (or $\mathrm{Hom}_A(M,-)$, respectively) for some A-module M, that is, there exists a functorial epimorphism $\mathrm{Hom}_A(-,M) \longrightarrow F \longrightarrow 0$, (or a functorial epimorphism $\mathrm{Hom}_A(M,-) \longrightarrow F \longrightarrow 0$, respectively).

We now characterise the finitely generated projective objects in our functor categories $\mathcal{F}un^{\mathrm{op}} A$ and $\mathcal{F}un\, A$.

6.5. Lemma. (a) *An object in $\mathcal{F}un^{op} A$ is finitely generated projective if and only if it is isomorphic to a functor of the form* $\mathrm{Hom}_A(-,M)$, *for M an A-module. Such a functor is indecomposable if and only if M is indecomposable.*

(b) *An object in $\mathcal{F}un\, A$ is finitely generated projective if and only if it is isomorphic to a functor of the form* $\mathrm{Hom}_A(M,-)$, *for M an A-module. Such a functor is indecomposable if and only if M is indecomposable.*

Proof. We only prove (a); the proof of (b) is similar. The projectivity of the finitely generated functor $\mathrm{Hom}_A(-,M)$ follows from (6.4). Conversely, let F be a finitely generated projective object in $\mathcal{F}un^{\mathrm{op}} A$, then there exists a functorial epimorphism $\varphi : \mathrm{Hom}_A(-,X) \longrightarrow F$, for some A-module X. Because F is projective, φ is a retraction and so there exists a functorial monomorphism $\psi : F \longrightarrow \mathrm{Hom}_A(-,X)$ such that $\varphi\psi = 1_F$. Let $\pi = \psi\varphi : \mathrm{Hom}_A(-,X) \longrightarrow F \longrightarrow \mathrm{Hom}_A(-,X)$ (thus, $F = \mathrm{Im}\,\pi$). By (6.2), there exists an endomorphism f of X such that $\pi = \mathrm{Hom}_A(-,f)$. Because π is an idempotent, we have $\mathrm{Hom}_A(-,f^2) = \mathrm{Hom}_A(-,f)^2 = \pi^2 = \pi = \mathrm{Hom}_A(-,f)$ thus $f^2 = f$, again by (6.2), that is, f is an idempotent. Consequently, $M = \mathrm{Im}\, f$ is a direct summand of X. Because $\mathrm{Hom}_A(-,M)$ is the image of $\mathrm{Hom}_A(-,f)$, we deduce that $F \cong \mathrm{Hom}_A(-,M)$. The same argument shows the last assertion. □

We now show that if M is an indecomposable module, the Hom functors $\mathrm{Hom}_A(-,M)$ and $\mathrm{Hom}_A(M,-)$ behave, in their respective categories, in a similar way to the finitely generated indecomposable projective modules over a finite dimensional algebra, in the sense that they have simple tops.

6.6. Lemma. *Let M be an indecomposable A-module.*

(a) *The functor* $\mathrm{rad}_A(-,M)$ *is the unique maximal subfunctor of the functor* $\mathrm{Hom}_A(-,M)$.

(b) *The functor* $\mathrm{rad}_A(M,-)$ *is the unique maximal subfunctor of the functor* $\mathrm{Hom}_A(M,-)$.

Proof. We only prove (a); the proof of (b) is similar. It suffices to show that any proper subfunctor F of $\mathrm{Hom}_A(-,M)$ is contained in $\mathrm{rad}_A(-,M)$, that is, for any indecomposable A-module N, we have $F(N) \subseteq \mathrm{rad}_A(N,M)$. If $N \not\cong M$, this follows from the fact that, by (A.3.5) of the Appendix, $\mathrm{rad}_A(N,M) = \mathrm{Hom}_A(N,M)$. Assume thus $N \cong M$ and let $f : M \to M$ belong to $F(M)$. By (6.2), $\mathrm{Hom}_A(-,f)$ maps $\mathrm{Hom}_A(-,M)$ to F, which is a proper subfunctor of $\mathrm{Hom}_A(-,M)$. Consequently, the functorial morphism $\mathrm{Hom}_A(-,f) : \mathrm{Hom}_A(-,M) \longrightarrow F \longrightarrow \mathrm{Hom}_A(-,M)$ is not an isomorphism. Hence neither is f and thus $f \in \mathrm{rad}_A(M,M)$. □

A nonzero functor is called **simple** if it has no nontrivial subfunctor.

Lemma 6.6 thus implies the following corollary.

6.7. Corollary. *Let M be an indecomposable A-module.*
(a) *The functor $S^M = \mathrm{Hom}_A(-,M)/\mathrm{rad}_A(-,M)$ is simple in $\mathcal{F}un^{op} A$.*
(b) *The functor $S_M = \mathrm{Hom}_A(M,-)/\mathrm{rad}_A(M,-)$ is simple in $\mathcal{F}un\, A$.* $\square$

In particular, $S^M(M) \cong S_M(M) \cong \mathrm{End}\, M/\mathrm{rad}\,\mathrm{End}\, M$ is a one-dimensional K-vector space (because the module M is indecomposable). By (6.2), this implies that $\mathrm{Hom}(\mathrm{Hom}_A(-,M), S^M)$ and $\mathrm{Hom}(\mathrm{Hom}_A(M,-), S_M)$ are also one-dimensional K-vector spaces and hence there exist nonzero functorial morphisms

$$\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S^M \quad \text{and} \quad \pi_M : \mathrm{Hom}_A(M,-) \longrightarrow S_M$$

that are uniquely determined up to a scalar multiple. Moreover, π^M and π_M are necessarily epimorphisms, because their targets are simple.

On the other hand, Corollary 6.7 also implies that if X is an indecomposable A-module not isomorphic to M, we have $S^M(X) = 0$ and $S_M(X) = 0$. Therefore the explicit expression of the functorial morphisms π^M and π_M follows from the proof of Yoneda's lemma, that is, if X is an indecomposable A-module, the morphisms $\pi^M(X) : \mathrm{Hom}_A(X,M) \longrightarrow S^M(X)$ and $\pi_M(X) : \mathrm{Hom}_A(M,X) \longrightarrow S_M(X)$ are both isomorphic to the canonical surjection $\mathrm{End}\, M \longrightarrow \mathrm{End}\, M/\mathrm{rad}\,\mathrm{End}\, M$ if $X \cong M$ and are zero otherwise.

Following (I.5.6), a functorial epimorphism $\varphi : F \to G$ in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$) is called **minimal** if, for each functorial morphism $\psi : H \to F$, the composite morphism $\varphi\psi$ is an epimorphism if and only if ψ is an epimorphism. A minimal functorial epimorphism $\varphi : F \to G$, with F projective, is called a **projective cover** of G.

An exact sequence $F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} G \longrightarrow 0$ in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un A$) is called a **projective presentation** of G. If, in addition, $\varphi_0 : F_0 \longrightarrow G$ is a projective cover and $\varphi_1 : F_1 \xrightarrow{\varphi_1} \mathrm{Im}\,\varphi_1$ is a projective cover, the sequence is called a **minimal projective presentation** of G.

We now prove the converse of Corollary 6.7, namely, we show that any simple contravariant (or covariant) functor is of the form described in (a) (or in (b), respectively) of the corollary.

6.8. Lemma. (a) *Let S be a simple object in $\mathcal{F}un^{op} A$. There exists, up to isomorphism, a unique indecomposable A-module M such that $S(M) \neq 0$. Further, $S \cong S^M$, the functorial morphism $\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S^M$ is a projective cover and $S(X) \neq 0$ if and only if M is isomorphic to a direct summand of X.*

(b) *Let S be a simple object in $\mathcal{F}un\, A$. There exists, up to isomorphism, a unique indecomposable A-module M such that $S(M) \neq 0$. Further, $S \cong S_M$, the functorial morphism $\pi_M : \mathrm{Hom}_A(M,-) \longrightarrow S_M$ is a projective cover, and $S(X) \neq 0$ if and only if M is isomorphic to a direct summand of X.*

Proof. We only prove (a); the proof of (b) is similar. Let S be a simple functor. We first note that, by Yoneda's lemma (6.1), $S(X) \neq 0$ for some A-module X if and only if there exists a nonzero functorial morphism $\pi^X : \mathrm{Hom}_A(-,X) \longrightarrow S$ that is necessarily an epimorphism, because S is simple. Because $S \neq 0$, there exists an indecomposable A-module M such that $S(M) \neq 0$. Let X be an arbitrary module such that $S(X) \neq 0$. We thus have functorial epimorphisms $\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S$ and $\pi^X : \mathrm{Hom}_A(-,X) \longrightarrow S$. By the projectivity of the functors $\mathrm{Hom}_A(-,M)$ and $\mathrm{Hom}_A(-,X)$ (see(6.4)), we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_A(-,M) \arrow[r, "\pi^M"] \arrow[d, "\mathrm{Hom}_A(-,f)"'] & S \arrow[r] \arrow[d, "1_S"] & 0 \\
\mathrm{Hom}_A(-,X) \arrow[r, "\pi^X"] \arrow[d, "\mathrm{Hom}_A(-,g)"'] & S \arrow[r] \arrow[d, "1_S"] & 0 \\
\mathrm{Hom}_A(-,M) \arrow[r, "\pi^M"] & S \arrow[r] & 0
\end{tikzcd}$$

where the existence of the morphisms $f : M \to X$ and $g : X \to M$ follows from (6.2). Because M is indecomposable, $\mathrm{End}\, M$ is local, hence $gf \in \mathrm{End}\, M$ must be nilpotent or invertible, by (I.4.6). However, if $(gf)^m = 0$ for some $m \geq 1$, we obtain $\pi^M = \pi^M \mathrm{Hom}_A(-,(gf)^m) = 0$, a contradiction. Hence gf is invertible so that f is a section and g is a retraction. Consequently, the functorial morphism $\mathrm{Hom}_A(-,g)$ is a retraction. This shows that $\pi^M : \mathrm{Hom}_A(-,M) \longrightarrow S$ is a projective cover. The uniqueness up to isomorphism of the indecomposable module M follows from the uniqueness up to isomorphism of the projective cover and (6.4). Finally, because, by (6.6), $\mathrm{Hom}_A(-,M)$ has $\mathrm{rad}(-,M)$ as unique maximal subfunctor, we infer the existence of a functorial isomorphism $S \cong \mathrm{Hom}_A(-,M)/\mathrm{rad}_A(-,M) = S^M$. $\square$

We have thus exhibited a bijective correspondence $M \mapsto S^M$ (or $M \mapsto S_M$) between the isomorphism classes of indecomposable A-modules and of simple objects in $\mathcal{F}un^{\mathrm{op}} A$ (or in $\mathcal{F}un\, A$, respectively). We now show that almost split morphisms in $\mathrm{mod}\, A$ correspond to projective presentations of these simple objects.

6.9. Lemma. (a) *Let N be an indecomposable A-module. A homomorphism $g : M \to N$ of A-modules is a right almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a projective presentation of S^N in $\mathcal{F}un^{op}A$.

(b) *Let L be an indecomposable A-module. A homomorphism $f : L \to M$ of A-modules is a left almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a projective presentation of S_L in $\mathcal{F}un\,A$.

Proof. We only prove (a); the proof of (b) is similar. Assume that g is right almost split. To prove that the induced sequence of functors is a projective presentation of S^N in $\mathcal{F}un^{\mathrm{op}}A$, it suffices, by (6.4), to prove it is exact, or equivalently, by (6.7), to prove that $\mathrm{Im}\,\mathrm{Hom}_A(-,g) = \mathrm{rad}_A(-,N)$. Thus, we must show that, for every indecomposable A-module X, $\mathrm{Im}\,\mathrm{Hom}_A(X,g) = \mathrm{rad}_A(X,N)$.

Let $h \in \mathrm{rad}_A(X,N)$. Then $h : X \to N$ is not an isomorphism. Because g is a right almost split morphism, there exists $k : X \to M$ such that $h = gk = \mathrm{Hom}_A(X,g)(k)$. Thus $\mathrm{rad}_A(X,N) \subseteq \mathrm{Im}\,\mathrm{Hom}_A(X,g)$. For the reverse inclusion, assume first $X \not\cong N$, then $\mathrm{rad}_A(X,N) = \mathrm{Hom}_A(X,N)$ and clearly $\mathrm{Im}\,\mathrm{Hom}_A(X,g) \subseteq \mathrm{Hom}_A(X,N)$; on the other hand, if $X \cong N$, this follows from the fact that g is not a retraction and (1.9). We have thus shown the necessity.

For the sufficiency, assume that the given sequence of functors is exact. We must show that g is right almost split. Suppose first that g is a retraction and $g' : N \to M$ is such that $gg' = 1_N$. Then, for any $h \in \mathrm{End}\,N$, we have $h = gg'h = \mathrm{Hom}_A(N,g)(g'h) \in \mathrm{Im}\,\mathrm{Hom}_A(N,g) = \mathrm{Ker}\,\pi^N_N$. This implies that $S^N(N) = 0$, a contradiction. Hence g is not a retraction. Let X be indecomposable, and $h : X \to N$ be a nonisomorphism, that is, $h \in \mathrm{rad}_A(X,N)$. Because the given sequence of functors is exact, evaluating these functors at X yields $\mathrm{rad}_A(X,N) = \mathrm{Ker}\,\pi^N_X = \mathrm{Im}\,\mathrm{Hom}_A(X,g)$. Hence there exists $k : X \to M$ such that $h = \mathrm{Hom}_A(X,g)(k) = gk$. Thus g is right almost split. □

Furthermore, minimal almost split morphisms in $\mathrm{mod}\,A$ correspond to minimal projective presentations of simple functors, as we show in the following lemma.

6.10. Lemma. (a) *Let N be an indecomposable A-module. A homomorphism $g : M \to N$ of A-modules is a right minimal almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective presentation of S^N in $\mathcal{F}un^{op}A$.

(b) *Let L be an indecomposable A-module. A homomorphism $f : L \to M$ of A-modules is a left minimal almost split morphism if and only if the induced sequence of functors*

$$\mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a minimal projective presentation of S_L in $\mathcal{F}un\,A$.

Proof. We only prove (a); the proof of (b) is similar. Assume that g is right minimal almost split. It follows from (6.9) that the induced sequence of functors is a projective presentation. We claim it is minimal, that is, by (6.6), $\mathrm{Hom}_A(-,g) : \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{rad}_A(-,N)$ is a projective cover. Let thus $\varphi : \mathrm{Hom}_A(-,X) \longrightarrow \mathrm{rad}_A(-,N)$ be a functorial epimorphism. It follows from (6.4) and (6.2) that there exist morphisms $u : M \to X$ and $v : X \to M$ such that we have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] \arrow[d, "\mathrm{Hom}_A(-,u)"'] & \mathrm{rad}_A(-,N) \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_A(-,X) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Hom}_A(-,v)"'] & \mathrm{rad}_A(-,N) \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] & \mathrm{rad}(-,N) \arrow[r] & 0
\end{tikzcd}$$

that is, $\mathrm{Hom}_A(-,g) \circ \mathrm{Hom}_A(-,v) \circ \mathrm{Hom}_A(-,u) = \mathrm{Hom}_A(-,g)$. By (6.2) again, $g(vu) = g$. Because g is right minimal, vu is an automorphism. Consequently, v is a retraction and therefore $\mathrm{Hom}_A(-,v)$ is a retraction. This shows that $\mathrm{Hom}_A(-,g) : \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{rad}_A(-,N)$ is a projective cover.

Conversely, if the shown sequence of functors is a minimal projective presentation, it follows from (6.9) that g is right almost split. We must show that it is right minimal. Assume $h : M \to M$ is such that $gh = g$. We have a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] \arrow[d, "\mathrm{Hom}_A(-,h)"'] & \mathrm{rad}_A(-,N) \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_A(-,M) \arrow[r, "\mathrm{Hom}_A(-,g)"] & \mathrm{rad}(-,N) \arrow[r] & 0
\end{tikzcd}$$

Because $\mathrm{Hom}_A(-,g)$ is a projective cover, $\mathrm{Hom}_A(-,h)$ is an isomorphism and hence so is h. □

We are now able to prove the main theorem of this section, which shows that almost split sequences in $\mathrm{mod}\,A$ correspond to minimal projective resolutions of simple functors in $\mathcal{F}un^{\mathrm{op}}A$ and in $\mathcal{F}un\,A$ defined in a usual way.

6.11. Theorem. (a) *Let N be an indecomposable A-module.*

(i) *N is projective, and $g: M \to N$ is right minimal almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}un^{op} A$.

(ii) *N is not projective, and the sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact and almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

(where $L \neq 0$) is a minimal projective resolution of S^N in $\mathcal{F}un^{op} A$.

(b) *Let L be an indecomposable A-module.*

(i) *L is injective, and $f : L \to M$ is left minimal almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

is a minimal projective resolution of S_L in $\mathcal{F}un\, A$.

(ii) *L is not injective, and the sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is exact and almost split if and only if the induced sequence of functors*

$$0 \longrightarrow \mathrm{Hom}_A(N,-) \xrightarrow{\mathrm{Hom}_A(g,-)} \mathrm{Hom}_A(M,-) \xrightarrow{\mathrm{Hom}_A(f,-)} \mathrm{Hom}_A(L,-) \xrightarrow{\pi_L} S_L \longrightarrow 0$$

(where $N \neq 0$) is a minimal projective resolution of S_L in $\mathcal{F}un\, A$.

Proof. We only prove (a); the proof of (b) is similar.

(i) Assume that N is projective, and $g : M \to N$ is right minimal almost split. By (3.5), g is a monomorphism with image equal to $\mathrm{rad}\, N$. By the left exactness of the Hom functor, $\mathrm{Hom}_A(-,g) : \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{Hom}_A(-,N)$ is a monomorphism. Thus, it follows from (6.10) that the induced sequence of functors

$$0 \longrightarrow \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$. Conversely, if the sequence of functors is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$, it follows from (6.10) that g is right minimal almost split. Evaluating the sequence of functors at A_A yields that g is a monomorphism. But, by the description of right minimal almost split morphisms in (3.1) and (3.2), this implies that N is projective.

(ii) Assume that N is not projective, and let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be an almost split sequence. By the left exactness of the Hom functor, we derive an exact sequence of projective functors

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N).$$

Because $g : M \to N$ is right minimal almost split, (6.10) yields that the induced sequence of functors

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \xrightarrow{\mathrm{Hom}_A(-,g)} \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0$$

is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$. Conversely, assume that the sequence of functors (where $L \neq 0$) is a minimal projective resolution of S^N in $\mathcal{F}un^{\mathrm{op}} A$. First, we claim that N is not projective. Indeed, if this were the case, then S^N has, by (a), a minimal projective resolution of the form

$$0 \longrightarrow \mathrm{Hom}_A(-,\mathrm{rad}\, N) \longrightarrow \mathrm{Hom}_A(-,N) \xrightarrow{\pi^N} S^N \longrightarrow 0,$$

where the first morphism is induced from the canonical inclusion of $\mathrm{rad}\, N$ into N. We thus have a short exact sequence of functors

$$0 \longrightarrow \mathrm{Hom}_A(-,L) \xrightarrow{\mathrm{Hom}_A(-,f)} \mathrm{Hom}_A(-,M) \longrightarrow \mathrm{Hom}_A(-,\mathrm{rad}\, N) \longrightarrow 0$$

that splits, because $\mathrm{Hom}_A(-,\mathrm{rad}\, N)$ is projective. In particular, the morphism $\mathrm{Hom}_A(-,f)$ is a section, a contradiction to the minimality of the given projective resolution. This shows our claim that N is not projective. In particular, N is not isomorphic to a direct summand of A_A hence, by (6.8), $S^N(A_A) = 0$. Evaluating the given projective resolution at A_A yields a short exact sequence of A-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

where, by (6.10), g is right minimal almost split. But this implies, by (1.13), that the sequence is almost split. $\square$

It is useful to observe that it follows from (6.11)(a) that, for any projective A-module P, there exists a functorial isomorphism $\mathrm{rad}_A(-,P) \cong \mathrm{Hom}_A(-,\mathrm{rad}\, P)$. Dually, for any injective A-module I, there exists a functorial isomorphism $\mathrm{rad}_A(I,-) \cong \mathrm{Hom}_A(I/\mathrm{soc}\, I,-)$.