Date due: Monday March 7, 2011. There are 12 questions on this assignment. Making genuine attempts at 5 of them will get an A.

1. (D&F 17.3, 4) Let V be the Klein 4-group and let $G = \mathrm{Aut}(V) \cong S_3$ act on V in the natural fashion. Prove that $H^1(G, V) = 0$. [Show that in the semidirect product $E = V \rtimes G$, G is the normalizer of a Sylow 3-subgroup of E. Apply Sylow's Theorem to show all complements to V in E are conjugate.]

2. Given a free presentation $1 \to R \to F \to G \to 1$ of a group G we have constructed a short exact sequence of $\mathbb{Z}G$-modules $0 \to R/R' \to \mathbb{Z}G^{d(F)} \to IG \to 0$ where $d(F)$ is the rank of the free group F. By using this sequence, give a proof of the rank formula $d(R) = 1 + |G|(d(F) - 1)$ when G is finite. [Assume that R is a free group – subgroups of free groups are free.]

3. Let M be a normal subgroup of a group E, write $G = E/M$ and suppose that M is generated as a normal subgroup of E by elements $m_1, m_2, \ldots, m_s$.
    (i) Show that $m_1 M', \ldots, m_s M'$ generate M/M' as a $\mathbb{Z}G$-module.
    (ii) Show that $m_1 - 1, \ldots, m_s - 1$ generate $\mathbb{Z}E \cdot IM = \mathbb{Z}E \cdot IM \cdot \mathbb{Z}E$ as a 2-sided ideal of $\mathbb{Z}E$.

4. Let $1 \to M \xrightarrow{i} E \xrightarrow{p} G \to 1$ be a group extension in which M is abelian, and let $s : G \to E$ be a section for p, that is, a mapping which satisfies $ps = 1_G$. Thus each element of E can be written in the form $i(m) \cdot s(x)$ with m and x uniquely determined. The multiplication in E determines a function $f : G \times G \to M$ by

$$s(x) \cdot s(x') = i f(x, x') \cdot s(xx'), \quad x, x' \in G.$$

   (i) Show that associativity of multiplication in E implies

$$x f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0, \quad \text{for all } x, y, z \in G.$$

   A function f satisfying this condition is called a *factor set*.
   (ii) Show that factor sets form a group under $(f_1 + f_2)(x, x') = f_1(x, x') + f_2(x, x')$.
   (iii) Show that if $g : G \to M$ is any function then the function which sends (x, y) to $g(xy) - g(x) - xg(y)$ is a factor set.
   (iv) Show that if $s, s' : G \to E$ are two sections and f, f' the corresponding factor sets, then there is a function $g : G \to M$ with $f'(x, y) = f(x, y) + g(xy) - g(x) - xg(y)$.
   [In fact the quotient of the group of factor sets by the factor sets of the form g is isomorphic to $H^2(G, M)$ and we have gone some way towards showing from this point of view that this group bijects with equivalence classes of extensions.]

5. Show that there are three equivalence classes of extensions

$$1 \to C_2 \to E \to C_2 \times C_2 \to 1$$

   in which E is isomorphic to D_8 and one equivalence class in which E is isomorphic to Q_8. Show that the sum of the Q_8 extension with any of the D_8 extensions is an extension in which the middle group is abelian.

6. (a) Let G be a group with a presentation $G = \langle g_1, \ldots, g_d \mid a_1, \ldots, a_r \rangle$ and suppose that the abelianisation G/G' is the direct sum of a free abelian group of rank s and a finite group. Show that $H_2(G, \mathbb{Z})$ can be generated by no more than $r - d + s$ elements.
   (b) Show that the braid group on three strings $B_3 = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$ has trivial Schur multiplier. Show that $H_2(\mathbb{Z}^n, \mathbb{Z})$ can be generated by at most $\binom{n}{2}$ elements.

7. By considering the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/|G|\mathbb{Z} \to 0$ show that if G is finite and $H_2(G, \mathbb{Z}) = 0$ then $H_2(G, \mathbb{Z}/|G|\mathbb{Z}) \cong G/G'$. (This suggests that it does not work to compute the Schur multiplier using a finite coefficient module.)

Questions on the Free Differential Calculus of R.H. Fox

8. Suppose that G is generated by elements $g_1, \ldots, g_n$, that is $G = \langle g_1, \ldots, g_n \rangle$, and let $d : G \to M$ be a derivation. Show that for each element $g \in G$ there exist elements $\lambda_i \in \mathbb{Z}G$ (which will depend on g) such that $d(g) = \sum \lambda_i d(g_i)$. Conclude that the $\mathbb{Z}$-linear span of $d(G)$ is the $\mathbb{Z}G$-submodule of M generated by $d(g_1), \ldots, d(g_n)$.

9. Apply the last question in the case of the derivation $d : G \to \mathbb{Z}G$ given by $d(g) = g - 1$, so that this defines $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}G$. Show that now for any other derivation $e : G \to N$ we also have $e(g) = \sum \lambda_i e(g_i)$, this equation holding in N.

10. Let F be the free group on generators $g_1, \ldots, g_n$ and consider $d : F \to \mathbb{Z}F$ given by $d(g) = g - 1$. Show that the elements λ_i considered in the last questions are now uniquely determined. We will denote the element $\lambda_i \in \mathbb{Z}F$ by $\frac{\partial g}{\partial g_i}$. Thus

$$d(g) = \sum_i \frac{\partial g}{\partial g_i} d(g_i).$$

   Show that
   (i) the mapping $\frac{\partial}{\partial g_i} : F \to \mathbb{Z}F$ is a derivation,
   (ii) $\frac{\partial g_j}{\partial g_i} = \delta_{ij}$.
   The properties (i) and (ii) in fact characterize $\frac{\partial}{\partial g_i}$. Demonstrate this by computing $\frac{\partial}{\partial x}(yx^{-1}yx^2)$.
   [The approach presented here is the 'free differential calculus' of R.H. Fox, and one of

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its uses can be found described in the book on knot theory by R.H. Crowell and R.H. Fox.]

11. Let $1 \to K \to E \to G \to 1$ be an extension of groups, with $K \leq E$. Show that if $k_1, \ldots, k_m$ generate K as a normal subgroup of E then $k_1 K', \ldots, k_m K'$ generate K/K' as a $\mathbb{Z}G$-module.

12. Let $1 \to R \to F \to G \to 1$ be a presentation of G, so

$$G = \langle g_1 \ldots g_n \mid r_1, \ldots, r_m \rangle,$$

where R is the normal subgroup of F generated by $r_1, \ldots, r_m$. Show that

$$r_j - 1 = \sum_i \frac{\partial r_j}{\partial g_i}(g_i - 1), \quad j = 1, \ldots, m.$$

Deduce that in the start of the resolution

$$\mathbb{Z}G^m \xrightarrow{\alpha} \mathbb{Z}G^n \longrightarrow \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

with R/R' below the left part and IG below the right part,

where the basis vectors of $\mathbb{Z}G^m$ are mapped to the generators $r_j R'$ of R/R', the map α has matrix $\left(\frac{\partial r_j}{\partial g_i} \right)_{i,j}$.

Evaluate this matrix in the case of the group given by the presentation

$$G = \langle x, y \mid x^4 = 1 = y^4, \; x^2 = y^2, \; yxy^{-1} = x^3 \rangle.$$

[One might call this the *Jacobian matrix*. It appears in the definition of the Alexander polynomial of a knot.]