Math 8246 **Homework 6** **PJW**

Date due: May 16, 2011 or sooner, either by email or in my mail box.

Notation: M^λ is the permutation module with the λ-tabloids as a basis. $S_F(n,r) = \mathrm{End}_{FS_r}(E^{\otimes r})$ where E is a vector space of dimension n over F.

1. Find a basis for the space of homomorphisms $\mathrm{Hom}_{FS_5}(M^{(3,2)}, M^{(2,1,1,1)})$. For each element θ in your basis, compute the effect of θ on the tabloid

$$\begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}.$$

2. (a) Let F be an infinite field, and suppose that $\rho : GL_1(F) \to GL_1(F)$ is a 1-dimensional polynomial representation of $GL_1(F)$. Show that ρ has the form $\rho(\lambda) = \lambda^n$ for some n. [Hint: Allowing λ to vary, compare $\rho(\lambda)^m$ and $\rho(\lambda^m)$, and use the fact that a polynomial which vanishes identically must be zero.]
(b) Exhibit a 1-dimensional representation of $GL_1(\mathbb{C})$ which is not rational (i.e. not of the form $\rho(\lambda) = f(\lambda)/g(\lambda)$ for polynomials f and g).

3. (a) Show that every subrepresentation of a polynomial representation is polynomial. [You may use without proof the fact that if $\rho : GL_n(F) \to GL(V)$ is given by matrices $\rho_{i,j}(g)$ with respect to one basis of V, and matrices $\sigma_{i,j}(g)$ with respect to another basis, then the $\rho_{i,j}$ are polynomial functions (homogeneous of degree r) if and only if the $\sigma_{i,j}$ are.]
(b) Show that if V_1, V_2 are invariant subspaces of a representation V of $GL_n(F)$, and that V_1 and V_2 are polynomial (homogeneous of degree r) then $V_1 + V_2$ is polynomial (homogeneous of degree r). Deduce that each finite dimensional representation V of $GL_n(F)$ has a unique largest subrepresentation which is polynomial, and also for each r a unique largest subrepresentation which is polynomial of degree r.

4. Show by example that the homomorphism $FGL(E) \to S_F(n,r)$ given by the representation of $GL(E)$ on $E^{\otimes r}$ need not be surjective if the field F is not infinite.

5. Let H be the subspace of $E^{\otimes r}$ spanned by the elements

$$e_{i_1} \otimes \cdots \otimes e_{i_r} - e_{i_{(1)\pi}} \otimes \cdots \otimes e_{i_{(r)\pi}}$$

as π ranges over the elements of S_r. The rth symmetric power of E is defined to be

$$S^r(E) := E^{\otimes r}/H.$$

The image of a tensor $e_{i_1} \otimes \cdots \otimes e_{i_r}$ in $S^r(E)$ may be written as a monomial $e_{i_1} \cdots e_{i_r}$. The degree r symmetric tensors $ST^r(E)$ are defined to be the fixed points of S_r in its action on $E^{\otimes r}$.
(a) Show that $ST^r(E)$ is a $FGL(E)$-submodule of $E^{\otimes r}$.

(b) Show that $ST^r(E)$ is spanned by tensors of the form

$$\sum_{\underline{j} \in \underline{i} S_r} e_{\underline{j}}$$

where the sum is over the S_r-orbit of multi-indices containing $\underline{i}$.

(c) When F has characteristic 0 or $p > r$ show that $E^{\otimes r} = ST^r(E) \oplus H$. Deduce that $ST^r(E) \cong S^r(E)$ as $GL(E)$-modules.

(d) Let F be a field of charactristic 2. In the particular case when $\dim E = 2$ and $r = 2$ show that $ST^2(E)$ has a 1-dimensional $FGL(E)$-submodule (hint: it is the space of skew-symmetric tensors) but that $S^2(E)$ has no 1-dimensional $FGL(E)$-submodule. (Hint: consider the action of matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ on $S^2(E)$.)

Deduce that $S^2(E)$ is not isomorphic to $ST^2(E)$ as $FGL(E)$-modules.

[In fact $S^r(E)$ is isomorphic to the dual (in a certain sense) of $ST^r(E)$. These two modules have the same composition factors - $\Lambda^2(E)$ and E in case $r = 2$.]