For example, if X is the Cayley graph of G with respect to a generating set S then the subset consisting of 1 and the edges $(1, s)$, $s \in S$, is a fundamental G-transversal. See Example 2.2.

For $G = 1$ the above argument shows the following.

2.7 Corollary. *If X is a connected graph then X has a maximal subtree. Any maximal subtree of X has vertex set all of VX.* ∎

3 Graphs of groups

We now introduce the main object of study for this chapter.

3.1 Definitions. By a *graph of groups* $(G(-), Y)$ we mean a connected graph $(Y, V, E, \bar{\iota}, \bar{\tau})$ together with a function $G(-)$ which assigns to each $v \in V$ a group $G(v)$, and to each edge $e \in E$ a distinguished subgroup $G(e)$ of $G(\bar{\iota}e)$ and an injective group homomorphism $t_e: G(e) \to G(\bar{\tau}e), g \mapsto g^{t_e}$. We call the $G(v), v \in V$, the *vertex groups*, the $G(e), e \in E$, the *edge groups*, and the t_e the *edge functions*.

In examples we sometimes depict this by labelling the vertices and edges of Y with the corresponding groups, see Example 3.3. ∎

3.2 General example. Let X be a G-graph such that $G \backslash X$ is connected and choose a fundamental G-transversal Y for X with subtree Y_0.

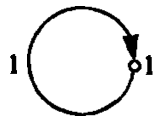
Since each element of X lies in the same G-orbit as a unique element of Y, for each $e \in EY$ there are unique $\bar{\iota}e, \bar{\tau}e \in VY$ which lie in the same G-orbits as $\iota e, \tau e$, respectively, and in fact $\bar{\iota}e = \iota e$. Using the incidence functions $\bar{\iota}, \bar{\tau}: EY \to VY$ we make Y into a graph, and clearly $Y \approx G \backslash X$. Notice Y_0 is simultaneously a maximal subtree of Y and a subtree of X. Observe that Y is not a subgraph of X unless $\bar{\tau}$ agrees with τ; in particular, an arbitrary maximal subtree of Y need not be a subgraph of X.

For each $e \in EY, \tau e$ and $\bar{\tau}e$ lie in the same G-orbit in EX, so we can choose an element t_e of G such that $t_e \bar{\tau}e = \tau e$; if $e \in EY_0$ then $\bar{\tau}e = \tau e$ and we take $t_e = 1$. We then call $(t_e | e \in EY)$ a *family of connecting elements.*

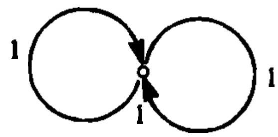
Now $G_e \subseteq G_{\iota e}$ *and* $G_e \subseteq G_{\tau e} = t_e G_{\bar{\tau}e} t_e^{-1}$ so there is an embedding $t_e: G_e \to G_{\bar{\tau}e}, g \mapsto g^{t_e} = t_e^{-1} g t_e$.

This data gives the graph of groups *associated to* X with respect to the fundamental G-transversal Y, the maximal subtree Y_0, and the family of connecting elements t_e. ∎

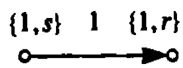
3.3 Specific examples. (i) In Example 2.2(i), (ii), the illustrated fundamental G-transversal, with connecting element s, gives the graph of groups



(ii) In Example 2.2(iii), (iv), the illustrated fundamental G-transversal, with connecting elements r, s gives the graph of groups



(iii) In Example 2.2(v), (vi), the illustrated fundamental G-transversal, with connecting elements 1, 1, gives the graph of groups



Thus a group acting on a graph with connected quotient graph determines a graph of groups. Conversely we now show how a graph of groups determines a group acting on a graph with connected quotient graph.

3.4 Definitions. Let $(G(-), Y)$ be a graph of groups as in Definition 3.1. Choose a maximal subtree Y_0 of Y, so $VY_0 = VY$ by Corollary 2.7. We define the associated *fundamental group* $\pi(G(-), Y, Y_0)$ to be the group presented with

generating set: $\{t_e \mid e \in E\} \vee \bigvee_{v \in V} G(v)$

relations:

the relations for $G(v)$, for each $v \in V$,

$t_e^{-1} g t_e = g^{t_e}$ for all $e \in E, g \in G(e) \subseteq G(\iota e)$, so $g^{t_e} \in G(\tau e)$,

$t_e = 1$, for all $e \in EY_0$.

Let $G = \pi(G(-), Y, Y_0)$.

In Corollary 7.5 we shall see that the $G(v)$ embed in G, and can be treated as subgroups.

To reduce the risk of confusion about the symbol t_e being used to denote an element of (the generating set of) G and a homomorphism, we make the convention that $(-)^{t_e}$ always refers to the homomorphism in this context.

Let T be the G-set presented with generating set Y, and relations saying that each $y \in Y$ is $G(y)$-stable. Then T has G-subsets $VT = GV$, $ET = GE$ and here $ET = T - VT$. It is straightforward to verify that there are well-defined G-maps $\iota, \tau: ET \to VT$ such that $\iota(ge) = g\bar{\iota}e$, $\tau(ge) = gt_e\bar{\tau}e$ for all $g \in G, e \in E$. This data then gives a G-graph T, which is in fact a tree, as will be seen in Theorem 7.6. We call T the associated *standard graph* or *standard tree*, denoted $T(G(-), Y, Y_0)$. It is straightforward to show that Y is a fundamental G-transversal in T, and for each $v \in V$,

$$\iota^{-1}(v) = \bigcup_{e \in \bar{\iota}^{-1}(v)} G(v)e \approx \bigvee_{e \in \bar{\iota}^{-1}(v)} G(v)/G(e)$$

$$\tau^{-1}(v) = \bigcup_{e \in \bar{\tau}^{-1}(v)} G(v)t_e^{-1}e \approx \bigvee_{e \in \bar{\tau}^{-1}(v)} G(v)/G(e)^{t_e}.$$

Thus in T, v has $\sum_{e \in \bar{\iota}^{-1}(v)} (G(v):G(e))$ edges going out and $\sum_{e \in \bar{\tau}^{-1}(v)} (G(v):t_e(G(e))$ edges going in. ∎

Notice that once one knows that the vertex groups embed in the fundamental group, it is a simple exercise to verify that a graph of groups can be recovered from the fundamental group acting on the standard graph.

In the next section we shall see that, conversely, starting from a group acting on a tree and forming the graph of groups we recover the group acting on the tree.

Hence the graphs of groups occurring in Example 3.3(i), (ii), (iii) have associated fundamental group and standard tree given in Example 2.2(ii), (iv), (vi), respectively.

We conclude this section with examples of special graphs of groups which occur throughout the sequel.

3.5 Examples. Let $(G(-), Y)$ be a graph of groups and Y_0 a maximal subtree of Y.

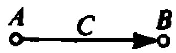
(i) Suppose $G(y) = 1$ for all $y \in Y$. Here $\pi(G(-), Y, Y_0)$ is denoted $\pi(Y, Y_0)$. It is the group presented on generators $\{t_e | e \in E\}$ with relations $t_e = 1$ for all $e \in EY_0$, so is a free group of rank $|EY - EY_0|$.

We have seen this situation in Example 3.3(i), (ii) and Example 2.2(ii), (iv) with Y consisting of one, two loops, respectively.

Since its isomorphism type is independent of the choice of Y_0, the group $\pi(Y, Y_0)$ is usually denoted $\pi(Y)$ called the *fundamental group* of Y. This will be discussed further in Definition 8.1.

(ii) Suppose Y has one edge e and two vertices $\iota e, \tau e$. Let $A = G(\iota e)$,

$B = G(\tau e)$, $C = G(e)$, so C is a subgroup of A and there is specified an embedding $C \to B, c \mapsto c^t$. The graph of groups is then depicted



Here $Y_0 = Y$, and the fundamental group is called the *free product of A and B amalgamating C*, denoted $A \underset{C}{*} B$; it is presented on generating set $A \vee B$ with relations saying $c = c^t$ for all $c \in C$, together with the relations of A and B. In the tree T, the vertices are of two sorts, with either $(A : C)$ edges going out, or $(B : C^t)$ edges going in.

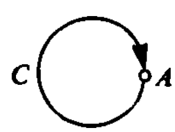
In the case $C = 1$, we write simply $A * B$, called the *free product* of A and B.

This is the situation in Example 3.3(iii) and Example 2.2(vi), and we can write $D_\infty = C_2 * C_2$.

(iii) Suppose Y is a tree and $G(e) = 1$ for all $e \in EY$. Then the fundamental group is the *free product* of the vertex groups $G(v), v \in VY$, denoted $\underset{v \in VY}{*} G(v)$.

(iv) Suppose $G(e) = 1$ for all $e \in EY$. Then the fundamental group is $\pi(Y, Y_0) * \underset{v \in VY}{*} G(v)$.

(v) Suppose Y has one edge e and one vertex $v = \iota e = \tau e$. Let $A = G(v)$, $C = G(e)$, so C is a subgroup of A and there is specified an embedding $C \to A, c \mapsto c^t$.



Here Y_0 consists of the single vertex, and the fundamental group is called the *HNN extension of A by t*: $C \to A$, denoted $A \underset{C}{*} t$; it is formed by adjoining to A an indeterminate t satisfying relations $t^{-1} c t = c^t$ for all $c \in C$. Every vertex of the tree T has $(A : C^t)$ edges going in, and $(A : C)$ edges going out.

If $C = 1$ we write $A * t$, so $A * t \approx A * C_\infty$.

We have seen the case $A = C = 1$ in Examples 3.3(i) and 2.2(ii).

A more complicated example arises by taking $A = \langle s \mid \varnothing \rangle, C = \langle s^3 \rangle$, and $t : C \to A$ with $(s^3)^t = s^2$. Here $G = \langle s, t \mid t^{-1} s^3 t = s^2 \rangle$ and T is a tree in which every vertex has two edges going in and three edges going out.

(vi) The fundamental group of any graph of groups can be obtained by successively performing one free product with amalgamation for each edge in the maximal subtree and then one HNN extension for each edge not in the maximal subtree. ∎

4 Groups acting on trees

Throughout this section let T be a G-tree.

4.1 Structure Theorem for groups acting on trees. *In the G-tree T choose a fundamental G-transversal Y with subtree Y_0 and denote the incidence functions by $\bar{\iota}, \bar{\tau}$; choose, for each $e \in EY$, $t_e \in G$ such that $t_e \bar{\tau} e = \tau e$, with $t_e = 1$ if $e \in EY_0$; and form the resulting graph of groups $(G(-), Y)$. Then G is naturally isomorphic to $\pi(G(-), Y, Y_0)$.*

Explicitly G has as a presentation:

(1) generating set: $\{t_e \mid e \in EY\} \vee \bigvee_{v \in VY} G_v$.

(2) relations:

the relations for G_v, for each $v \in VY$;

$t_e^{-1} g t_e = g^{t_e}$, for all $e \in EY, g \in G_e \subseteq G_{\bar{\iota} e}$ so $g^{t_e} \in G_{\bar{\tau} e}$;

$t_e = 1$, for all $e \in EY_0$.

Proof. Let us consider any $v \in VY$, and analyze the neighbours of v in T.

Consider any edge of T incident to v and express it in the form ge with $g \in G$, $e \in EY$.

If $\iota ge = v$ then $v = \iota ge = g\iota e = g\bar{\iota}e$, and, as Y is a G-transversal, $\bar{\iota}e = v$, $g \in G_v$ and $\tau ge = g\tau e = gt_e\bar{\tau}e$.

If $\tau ge = v$ then $v = \tau ge = g\tau e = gt_e\bar{\tau}e$, and, as Y is a G-tranversal, $\bar{\tau}e = v$ and $gt_e \in G_v$, so we can write $g = ht_e^{-1}$ with $h \in G_v$, and $\iota ge = g\bar{\iota}e = ht_e^{-1}\bar{\iota}e$.

Conversely, all edges of T constructed in this way are incident to v.

We can summarize this by saying that the paths of length 1 in T starting at v are the sequences of the form $v, gt_e^{\frac{1}{2}(\varepsilon-1)}e^\varepsilon, gt_e^\varepsilon w$ where v, e^ε, w is a path in Y, and $g \in G_v$.

Let $P = \pi(G(-), Y, Y_0)$, so P has presentation (1), (2). Since (1) is a subfamily of G and all the relations (2) hold in G, there is a natural homomorphism $P \to G$, and we wish to show it is an isomorphism.

Since G acts on T, the map $P \to G$ induces a P-action on T. Consider the subset PY of T. By the preceding paragraph, all edges of T incident to Y lie in PY, and so do their vertices; hence all edges of T incident to PY lie in PY and so do their vertices. Since T is connected it follows that $PY = T$. Choose any vertex v_0 of Y. For any $g \in G$, we have $gv_0 \in T = PY$ so $gv_0 = pv$ for some $p \in P$, $v \in Y$. But Y is a G-transversal so $v = v_0$. Since G_{v_0} is in the image of P, we see $P \to G$ is surjective.

Consider any $p \in P$. We claim we can choose a path $v_0, e_1^{\varepsilon_1}, v_1, e_2^{\varepsilon_2}$,

$v_2, \ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n = v_0$ in Y, and elements $g_i \in G_{v_i}$, $i \in [0, n]$, such that

(3) $\qquad p = g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n} g_n.$

This is achieved by expressing p as a product of the given generators and their inverses, then using the relations for the G_v to collect together generators from the same G_v into single expressions, and finally inserting 1's as dictated by paths in the maximal subtree Y_0 to obtain an expression as in (3).

It is straightforward to check that

(4) $$v_0, g_0 t_{e_1}^{\frac{1}{2}(\varepsilon_1 - 1)} e_1^{\varepsilon_1}, g_0 t_{e_1}^{\varepsilon_1} v_1, g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\frac{1}{2}(\varepsilon_2 \quad 1)} e_2^{\varepsilon_2},$$
$$g_0 t_1^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} v_2, \ldots, g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} \cdots g_{n-1} t_{e_n}^{\frac{1}{2}(\varepsilon_n - 1)} e_n^{\varepsilon_n},$$
$$g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} \cdots g_{n-1} t_{e_n}^{\varepsilon_n} v_n = p v_n$$

is then a path in T. Notice that, as P acts via the map $P \to G$, it is irrelevant whether the expressions are considered as representing elements of P or G.

We shall show by induction on n that if p is mapped to 1 in G then $p = 1$ in P. Since the composite $G_{v_0} \to P \to G$ is the inclusion map, we may assume $n \geqslant 1$. Since T is a tree and $n \geqslant 1$, the path (4) is not reduced, and for some $i \in [1, n-1]$ the ith edge and the $(i+1)$th edge are inverse to each other. It follows that $\varepsilon_{i+1} = -\varepsilon_i$ and $t_{e_i}^{\frac{1}{2}(\varepsilon_i - 1)} e_i = t_{e_i}^{\varepsilon_i} g_i t_{e_{i+1}}^{\frac{1}{2}(\varepsilon_{i+1} - 1)} e_{i+1}$. Since Y is a G-transversal in $T, e_{i+1} = e_i$ and $t_{e_i}^{\frac{1}{2}(\varepsilon_i + 1)} g_i t_{e_i}^{-\frac{1}{2}(\varepsilon_i + 1)} e_i = e_i$. Thus we have two generators $h = t_{e_i}^{\frac{1}{2}(\varepsilon_i + 1)} g_i t_{e_i}^{-\frac{1}{2}(\varepsilon_i + 1)} \in G_{e_i} \subseteq G_{\bar{\iota} e_i}$ and $h^{t_{e_i}} = t_{e_i}^{\frac{1}{2}(\varepsilon_i - 1)} g_i t_{e_i}^{-\frac{1}{2}(\varepsilon_i - 1)} \in G_{\bar{\tau} e_i}$, and, by (2), $t_{e_i}^{-1} h t_{e_i} = h^{t_{e_i}}$ in P.

If $\varepsilon_i = 1$ then $\bar{\iota} e_i = v_{i-1} = v_{i+1}$, $\bar{\tau} e_i = v_i$ so $h \in G_{\bar{\iota} e_i} = G_{v_{i-1}}$, and $h^{t_{e_i}} = g_i$ so $t_{e_i}^{-1} h t_{e_i} = g_i$ in P. Hence in (3) we can replace $g_{i-1} t_{e_i} g_i t_{e_i}^{-1} g_{i+1}$ with the single generator $g_{i-1} h g_{i+1} \in G_{v_{i+1}}$ and omit e_i, v_i, e_i^{-1} from the path in Y, and so reduce n by 2.

Similarly, if $\varepsilon_i = -1$ then $\bar{\iota} e_i = v_i, \bar{\tau} e_i = v_{i-1} = v_{i+1}$, so $h^{t_{e_i}} \in G_{\bar{\tau} e_i} = G_{v_{i-1}}$, and $h = g_i$ so $h^{t_{e_i}} = t_{e_i}^{-1} g_i t_{e_i}$ in P. Hence in (3) we can replace $g_{i-1} t_{e_i}^{-1} g_i t_{e_i} g_{i+1}$ with the single generator $g_{i-1} h^{t_{e_i}} g_{i+1} \in G_{v_{i-1}}$ and omit e_i^{-1}, v_i, e_i from the path in Y, and so reduce n by 2.

It follows by induction on n that $p = 1$, so $P \to G$ is injective, and G has the desired presentation. ∎

The case of a free action is particularly interesting.

4.2 Corollary. *If G acts freely on T then G is a free group, in fact, $G \approx \pi(G \backslash T)$.* ∎

Since trivial vertex groups correspond to free groups, there is a type of duality between vertex groups and free groups. We now state some results at the two extremes, for which we require the following observation.

4.3 Lemma. *If N is a normal subgroup of G then $N\backslash T$ is a connected G/N-graph and each $Nt \in N\backslash T$ has stabilizer $(G/N)_{Nt} = NG_t/N$.* ∎

4.4 Proposition. *If N is the subgroup of G generated by the G_v, $v \in VT$, then N is normal and G/N is free. Moreover, $N\backslash T$ is a G/N-free G/N-tree and $G/N \approx \pi(G\backslash T)$.*

Proof. It follows easily from the Structure Theorem 4.1 that $G/N \approx \pi(G\backslash T)$.

We now apply this with N in place of G. Since N is generated by the vertex stabilizers $N_v = G_v$, $v \in VT$, we see that $\pi(N\backslash T) \approx N/N = 1$. Hence $N\backslash T$ is a tree. By Lemma 4.3, G/N acts freely on $N\backslash VT$, and hence on $N\backslash T$, so $G/N \approx \pi(G\backslash T)$. ∎

4.5 Proposition. *If a subgroup H of G does not meet any vertex stabilizer then H acts freely on T, so H is free. For example, if H is torsion-free and the vertex stabilizers are torsion groups then H is free.* ∎

4.6 Proposition. *If $G \to A$ is a homomorphism of groups which is injective on each vertex stabilizer then the kernel N is free. In fact $N \approx \pi(X)$, where X is the connected G/N-graph $N\backslash T$.* ∎

If the homomorphism $G \to A$ is surjective then X is a connected A-graph. In Theorem 9.2 we shall see that all group actions on connected graphs can be realized in this way.

The Structure Theorem 4.1 suggests that there are only limited possibilities for a group to act on a tree if the group has certain special properties such as being finite, cyclic, soluble, free, etc. The finite case will be of great importance in Chapters 3 and 4, and it will be useful to know the following.

4.7 Proposition. *Let v be a vertex of T. Then G stabilizes a vertex of T if and only if there is an integer N such that the distance from v to each element of Gv is at most N.*

Proof. If G stabilizes a vertex v_0 of T, and the T-geodesic p from v to v_0 has length n, then for each $g \in G$ we have a path p, gp^{-1} of length $2n$ from v to gv.

Conversely, suppose there is an integer N such that for each $g \in G$ the T-geodesic from v to gv has length at most N. Let T' be the subtree generated by Gv.

It is easy to see that T' is a G-subtree of T and no reduced path has length greater than $2N$.

If T' has at most one edge then every element of T' is G-stable, and we have the desired G-stable vertex. Thus we may assume that T' has at least two edges, so some vertex of T' has valency at least two. Now delete from T' all vertices of valency one, and their incident edges. This leaves a G-subtree T'' in which no reduced path has length greater than $2N-2$. By induction, G stabilizes a vertex. ∎

Let us note the cases where Gv is finite, and where G is finite.

4.8 Corollary. *If there is a finite G-orbit in VT then G stabilizes a vertex of T.* ∎

4.9 Corollary. *A finite group acting on a tree must stabilize a vertex.* ∎

At this stage it is convenient to introduce some very useful terminology.

4.10 Definitions. Let e, f be edges of T and v a vertex of T.

We say that *e points to v* if e is the first edge in the T-geodesic from ιe to v, or equivalently, τe and v lie in the same component of $T - \{e\}$. Otherwise, *e points away from v.*

Consider a reduced path $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ in T with $e_1 = e, e_n = f$; it is unique unless $e = f$ and $n = 1$. If $\varepsilon_1 = \varepsilon_n$ we say that *e and f point in the same direction*; otherwise *e and f point in opposite directions.* We define a partial order $\geqslant$ on $EX^{\pm 1}$ by setting $e_1^{\varepsilon_1} \geqslant e_n^{\varepsilon_n}$.

An element g of G is said to *translate e* if ge and e are distinct and point in the same direction, that is, $ge > e$ or $e > ge$.

By an *infinite path* in X we mean a sequence $v_0, e_1^{\varepsilon_1}, v_1, \ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n, \ldots$ where for each $n \geqslant 0$, $v_n \in VX$, and for each $n \geqslant 1$, $e_n^{\varepsilon_n} \in EX^{\pm 1}$, $\iota e_n^{\varepsilon_n} = v_{n-1}$, $\tau e_n^{\varepsilon_n} = v_n$.

By a *doubly infinite path* in X we mean a sequence $\ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n, \ldots$ where for each $n \in \mathbb{Z}$, $v_n \in VX$, $e_n^{\varepsilon_n} \in EX^{\pm 1}$, $\iota e_n^{\varepsilon_n} = v_{n-1}$, $\tau e_n^{\varepsilon_n} = v_n$. ∎

4.11 Proposition. *For any $g \in G$ the following are equivalent:*

(*a*) *g does not stabilize a vertex of T.*

(*b*) *g translates an edge of T.*

(c) *g acts by translation on a subtree of T homeomorphic to* $\mathbb{R}$. *In this event g has infinite order.*

Proof. (a)⇒(c). Choose a vertex v of T in such a way that the T-geodesic $p = e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ from v to gv is as short as possible. Assume (a) holds so $n \geqslant 1$. Notice that $e_n^{-\varepsilon_n} \neq ge_1^{\varepsilon_1}$, for if $e_n^{-\varepsilon_n} = ge_1^{\varepsilon_1}$ then $n \geqslant 2$ and $e_2^{\varepsilon_2}, \ldots, e_{n-1}^{\varepsilon_{n-1}}$ is the T-geodesic from $\iota e_2^{\varepsilon_2}$ to $\tau e_{n-1}^{\varepsilon_{n-1}} = \iota e_n^{\varepsilon_n} = g\tau e_1^{\varepsilon_1} = g\iota e_2^{\varepsilon_2}$ which contradicts the minimality of n. Hence by concatenating the $g^m p$, $m \in \mathbb{Z}$, we can construct a reduced doubly infinite path which gives a subtree homeomorphic to $\mathbb{R}$, on which g acts by translation by n. It is not difficult to see that g acts by translating a fundamental G-transversal containing p.

(c)⇒(b)⇒(a) is clear. ∎

4.12 Theorem. *Exactly one of the following holds:*

(a) *G stabilizes a vertex of T.*

(b) *There is a reduced infinite path* $v_0, e_1^{\varepsilon_1}, v_1, e_2^{\varepsilon_2}, \ldots$ *in T such that* $G_{v_0} \subseteq G_{v_1} \subseteq \cdots$, $G = \bigcup_{n \geqslant 0} G_{v_n} = \bigcup_{n \geqslant 1} G_{e_n}$ *and for all* $n \geqslant 1, G \neq G_{e_n}$.

(c) *Some element of G translates some edge e of T, and then for* $C = G_e$, *either* $G = B \underset{C}{*} D$ *with* $B \neq C \neq D$ *or* $G = B \underset{C}{*} x$.

Proof. It is an easy matter to verify that (a), (b), (c) are pairwise incompatible, and we wish to show that at least one of them holds.

Suppose that (a) and (c) fail and consider any vertex v.

It follows from the failure of (a) that $G \neq G_v$. Consider any $g \in G - G_v$. Since (c) fails, it follows from Proposition 4.11 (a)⇒(b) that g stabilizes some vertex $w \neq v$ of T, and we may choose w as close as possible to v. Let e^ε be the first edge in the T-geodesic p from v to w. Then p, gp^{-1} is reduced, and hence $e^\varepsilon > ge^{-\varepsilon}$.

We claim that e^ε is independent of the choice of g. Suppose that $e'^{\varepsilon'}$ is the first edge in the geodesic from v to a vertex w' stabilized by $g' \in G - G_v$. Then $e'^{\varepsilon'} > g'e'^{-\varepsilon'}$. If $e^\varepsilon \neq e'^{\varepsilon'}$ then $e^{-\varepsilon}, e'^{\varepsilon'}$ is a reduced path so $e^{-\varepsilon} > e'^{\varepsilon'}$, and thus $g'e^{-\varepsilon} > g'e'^{\varepsilon'} > e'^{-\varepsilon'} > e^\varepsilon > ge^{-\varepsilon}$, which means that gg'^{-1} translates $g'e^{-\varepsilon}$, contradicting the failure of (c). Hence, $e^\varepsilon = e'^{\varepsilon'}$ as desired.

Hence, τe^ε is uniquely determined by v, and we denote it by $\phi(v)$. This gives a well-defined map $\phi: VT \to VT$.

It is clear that ϕ is a G-map, so $G_v \subseteq G_{\phi(v)}$.

By choosing $g \in G - G_{\phi(v)}$, we see $\phi^2(v) \neq v$; hence $v, \phi(v), \phi^2(v), \ldots, \phi^n(v), \ldots$ is the sequence of vertices in a reduced infinite path.

By using the same g, w to find as many of $\phi(v), \phi^2(v), \ldots, \phi^n(v)$ as possible, we deduce that $w = \phi^n(v)$ for some n. Hence $g \in \bigcup_{n \geqslant 0} G_{\phi^n(v)}$. It follows that $\bigcup_{n \geqslant 0} G_{\phi^n(v)} = G$, and (b) holds.

Thus (a), (b) or (c) holds.

Finally, suppose that (c) holds. Contracting all edges not in Ge yields a G-tree with exactly one edge orbit and with an edge e translated by an element of G, so no vertex is stabilized by G. By the Structure Theorem 4.1, either $G = B \underset{C}{*} D$ with $B \neq C \neq D$, or $G = B \underset{C}{*} x$. ∎

In case (c) somewhat more can be said.

4.13 Proposition. *If some element of G translates some edge of T then there is a unique minimal G-subtree T' of T and ET' consists of all edges translated by elements of G. If G is finitely generated then $G \backslash T'$ is finite.*

Proof. Let E' be the set of edges of T translated by elements of G, and let T' be the subgraph of T with edge set E'. By hypothesis, T' is non-empty; we claim T' is connected and hence is a subtree.

Suppose T_1, T_2 are subtrees of T homeomorphic to $\mathbb{R}$ on which elements g_1, g_2 act by translation, respectively. To show that T' is connected, it suffices to show that each edge in the path p joining T_2 to T_1 lies in E', so we may assume p is nonempty. By replacing g_1 and/or g_2 with its inverse if necessary, we may choose paths p_1, p_2 in T_1, T_2, respectively, so as to have the situation illustrated in Fig. I.2(i). Here each edge of p is translated by $g_1 g_2$ so it lies in E'; see Fig. I.2(ii). Thus T' is a subtree.

Clearly, E' is a G-subset of ET, so T' is a G-subtree of T.

If $g \in G$ translates an edge, then we have a $\langle g \rangle$-subtree T_g of T homeomorphic to $\mathbb{R}$, and $\langle g \rangle$ acts on T by translating T_g. It is then easy to see that T_g is the unique minimal $\langle g \rangle$-subtree of T, so lies in every G-subtree of T. Hence, T' is the unique minimal G-subtree of T.

Now suppose G has a finite generating set S. Let v be a vertex of T'

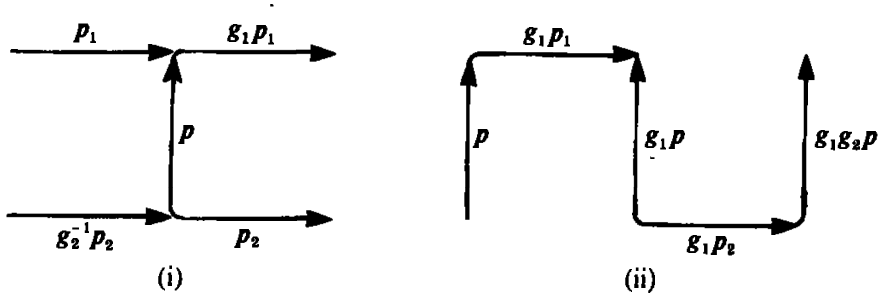


Fig. I.2

and let Y be the subtree of T' generated by $Sv \cup \{v\}$. The union of the $gY, g \in G$, gives a connected G-subgraph of T' which, by minimality, must be all of T'. Since Y is finite, T' is G-finite. ∎

5 Trees for certain automorphism groups

In this section we discuss trees which are acted on by the automorphism group of the free object of rank two in each of the following categories: abelian groups, free groups, vector spaces over principal valuated fields, commutative algebras over a field, and associative algebras over a field.

5.1 Notation. For any commutative ring R and positive integer n, $\mathrm{GL}_n(R)$ denotes the group of all invertible $n \times n$ matrices with entries in R. The centre of $\mathrm{GL}_n(R)$ is the group of invertible scalar matrices, and this can be identified with $UR = \mathrm{GL}_1(R)$, the group of units of R; the quotient group is denoted $\mathrm{PGL}_n(R)$. An $n \times n$ matrix A over R lies in $\mathrm{GL}_n(R)$ if and only if the determinant $\det A$ lies in UR. The subgroup of $n \times n$ matrices with determinant 1 is denoted $\mathrm{SL}_n(R)$, and the centre lies in the centre of $\mathrm{GL}_n(R)$ and the quotient group is denoted $\mathrm{PSL}_n(R)$. ∎

5.2 Examples. (i) Let $G = \mathrm{GL}_2(\mathbb{Z})$. There is a classical G-action on the upper half-plane $\mathscr{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$, with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acting on $z = x + iy$ to give

$$gz = \frac{az + b}{cz + d}$$

if $ad - bc = 1$, and

$$gz = \frac{a\bar{z} + b}{c\bar{z} + d}$$

if $ad - bc = -1$, where $\bar{z} = x - iy$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x + iy) = \frac{(ax + b)(cx + d) + acy^2 + iy}{(cx + d)^2 + c^2 y^2}.$$

Let $Y = \{\cos\theta + i \sin\theta \mid \pi/3 \leqslant \theta \leqslant \pi/2\}$, a subset of $\mathscr{H}$ which can be viewed as the geometric realization of a graph with one edge e having $\iota e = i, \tau e = \frac{1}{2}(1 + i\sqrt{3})$. Let $T = GY$, a G-subset of $\mathscr{H}$, which we shall see is

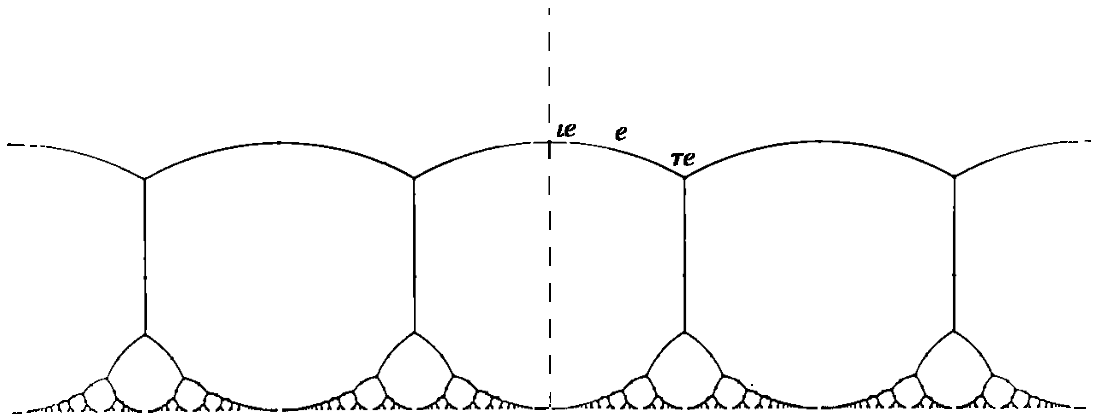


Fig. I.3

a geometric realization of a G-tree with fundamental G-transversal Y. See Fig. I.3.

A straightforward calculation shows that

$$G_{\iota e}=\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm\begin{pmatrix}0&-1\\1&0\end{pmatrix},\pm\begin{pmatrix}1&0\\0&-1\end{pmatrix},\pm\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$$
$$=D_4,$$

$$G_{\tau e}=\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm\begin{pmatrix}1&-1\\1&0\end{pmatrix},\pm\begin{pmatrix}0&1\\-1&1\end{pmatrix},\right.$$
$$\left.\pm\begin{pmatrix}0&1\\1&0\end{pmatrix},\pm\begin{pmatrix}1&0\\1&-1\end{pmatrix},\pm\begin{pmatrix}-1&1\\0&1\end{pmatrix}\right\}=D_6,$$

$$G_e=\left\{\pm\begin{pmatrix}1&0\\0&1\end{pmatrix},\pm\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}=D_2.$$

By the Euclidean algorithm, any 2×2 matrix over $\mathbb{Z}$ can be transformed to an upper triangular matrix using the operations of interchanging two rows and adding or subtracting the second row from the first; these operations correspond to left multiplication by

$$t=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad s=\begin{pmatrix}1&1\\0&1\end{pmatrix}\quad\text{and}\quad s^{-1}=\begin{pmatrix}1&-1\\0&1\end{pmatrix},$$

respectively. It follows that $\mathrm{GL}_2(\mathbb{Z})$ is generated by t,s and

$$r=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Notice that $t\in G_e$, $r\in G_{\iota e}$,

$$srt=\begin{pmatrix}-1&1\\-1&0\end{pmatrix}\in G_{\tau e},$$

so $G_{\iota e}\cup G_{\tau e}$ generates G. Since G permutes the components of T, we see that the component of T containing Y is closed under the action of $G_{\iota e} \cup G_{\tau e}$ so is closed under the action of G, so contains $GY = T$. Thus T is connected.

It remains to verify that T has no simple closed curves. Consider any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$$

and any $z = x + iy \in Y$. We claim that if $gz \in Y$ then either each point of Y is g-stable, or z is a g-stable endpoint of Y, and thus T is the geometric realization of a G-graph. We claim further that if $\mathrm{Re}(gz) = 0$ then $gz = \iota e$. There are two cases.

Consider first the case where $c^2 \neq d^2$.

Using the fact that $|z| = 1$ one can verify that $|(c^2 - d^2)gz - (ac - bd)| = 1$, that is, gz is on a circle with centre $(ac - bd)/(c^2 - d^2)$ and radius $1/|c^2 - d^2|$.

If $gz \in Y$, then $\mathrm{Im}(gz) \geqslant \sqrt{3}/2 \geqslant \frac{1}{2}$ so $|c^2 - d^2| \leqslant 2$ so $|c^2 - d^2| = 1$. Similarly the centre is limited to 0 or 1 so $ac - bd = 0$ or 1, and it can be verified that either each point of Y is g-stable, or z is a g-stable endpoint of Y. If $\mathrm{Re}(gz) = 0$ then the circle meets the upper half of the imaginary axis and this forces $ac - bd = 0$, from which it follows that each point of Y is g-stable, so $z = gz = \iota e$.

Now suppose that $c^2 = d^2$.

Then $c^2 = d^2 = 1$ and a, b have opposite parity. It can be shown that $2\,\mathrm{Re}(gz) = ac + bd$, an odd integer, so nonzero.

If $gz \in Y$ then $ac + bd = 1$ and $gz = \tau e$, and it follows that τe is g-stable.

Now suppose there is a simple closed path in T. Then there is a path in T from ιe to τe which does not use e. The only edges incident to ιe are e and the edge

$$e' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e.$$

So there is a path from $\tau e'$ to τe which does not pass through $\iota e = i$, but by continuity considerations it must pass through the imaginary axis, a contradiction. Thus T is a G-tree and we deduce $\mathrm{GL}_2(\mathbb{Z}) = D_4 \underset{D_2}{*} D_6$.

It is not difficult to deduce the presentation

$$\mathrm{GL}_2(\mathbb{Z}) = \langle q, r, t \mid q^2, r^2, t^2, (tr)^4, (tr)^2(tq)^3 \rangle$$

where

$$q = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following three groups also act on T with Y as fundamental transversal, and so have the descriptions indicated.

(ii) $\mathrm{PGL}_2(\mathbb{Z}) = D_2 \underset{D_1}{*} D_3$, where

$$D_2 = \left\{\left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\right\},\right.$$
$$\left.\left\{\pm\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right\}\right\},$$
$$D_3 = \left\{\left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}0 & 1\\-1 & 1\end{pmatrix}\right\},\right.$$
$$\left.\left\{\pm\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}1 & 0\\1 & -1\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}-1 & 1\\0 & 1\end{pmatrix}\right\}\right\},$$
$$D_1 = \left\{\left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right\}\right\}.$$

(iii) $\mathrm{SL}_2(\mathbb{Z}) = C_4 \underset{C_2}{*} C_6$, where

$$C_4 = \left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \pm\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\right\},$$
$$C_6 = \left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}, \pm\begin{pmatrix}0 & 1\\-1 & 1\end{pmatrix}\right\},$$
$$C_2 = \left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}.$$

(iv) $\mathrm{PSL}_2(\mathbb{Z}) = C_2 * C_3$, where

$$C_2 = \left\{\left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\right\}\right\},$$
$$C_3 = \left\{\left\{\pm\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}\right\}, \left\{\pm\begin{pmatrix}0 & 1\\-1 & 1\end{pmatrix}\right\}\right\}.$$

(v) Denote the automorphism group of F_2 by $\mathrm{Aut}(F_2)$. Let F_2 be free on x, y so an automorphism ϕ can be represented by the ordered pair $(\phi x, \phi y)$. We identify F_2 with the group of inner automorphisms, with F_2 acting by left conjugation.

It can be shown that F_2 is generated by $q = (xy, y^{-1})$, $r = (x, y^{-1})$, $t = (y, x)$; see Lyndon and Schupp (1977), Proposition I.4.1. One can compute that $1 = q^2 = r^2 = t^2 = (tr)^4$ and $(tr)^2(tq)^3 = xy$.

Every automorphism of F_2 induces an automorphism on the abelianization $\mathbb{Z}^2$, so there is a homomorphism from $\mathrm{Aut}(F_2)$ to $\mathrm{GL}_2(\mathbb{Z})$, and the preceding paragraph, together with (i), shows that $\mathrm{Aut}(F_2) \to \mathrm{GL}_2(\mathbb{Z})$ is surjective with kernel F_2. Thus $\mathrm{Aut}(F_2) = A \underset{C}{*} B$, where A, B, C are extensions of F_2 by D_4, D_6 and D_2, respectively. See Example IV.1.13 for a further discussion of A, B, C. ∎

5.3 Examples. (i) Let R be a principal valuation ring, K its field of fractions, k the residue field, and t a uniformizer, so $R/tR = k$. For example, R can be the ring of formal power series $k[[t]]$, and then K is the field $k((t))$ of Laurent series.

Let U denote the group of units of R, K^* the group of units of K, and $G = \mathrm{GL}_2(K)$, and view K^* as the centre of G.

We shall construct a G-tree.

Let K^2 be the set of column vectors of length two over K, so K^2 is a G-module under matrix multiplication.

By an *R-lattice* P in K^2 we mean a free R-submodule of K^2 of rank 2, that is, P is generated by a K-basis of K^2. Let $\mathscr{L}$ denote the set of all R-lattices in K^2, so $\mathscr{L}$ is a G-set.

For $P \in \mathscr{L}$ we write $[P] = \{t^i P \mid i \in \mathbb{Z}\} = K^* P$. Then $K^* \backslash \mathscr{L} = \{[P] \mid P \in \mathscr{L}\}$ is again a G-set, since K^* is the centre of G.

If $P, Q \in \mathscr{L}$ then by the theory of matrices over principal ideal domains there exists an R-basis b, b' of P and integers m, n such that $t^m b, t^n b'$ is an R-basis of Q, and then $\{m, n\}$ is independent of the choice of b, b'. We define $d(P, Q) = |m - n|$, and also $d([P], [Q]) = d(P, Q) = |m - n|$, which is easily seen to be well-defined.

Notice $[P] = \{Q \in \mathscr{L} \mid d(P, Q) = 0\}$.

Let $E = \{(p, q) \mid p, q \in K^* \backslash \mathscr{L}$ with $d(p, q) = 1\}$. We claim that E is the edge set of a G-tree T. To construct T we take vertex set $V = \{p, \{p, q\} \mid (p, q) \in E\}$, and incidence functions $\iota(p, q) = p$, $\tau(p, q) = \{p, q\}$. It is clear that T is a G-graph.

If $p, q \in K^* \backslash \mathscr{L}$ then the paths in T from p to q can be identified with the sequences $p = p_0, p_1, \ldots, p_n = q$ such that $d(p_{i-1}, p_i) = 1$ for all $i \in [1, n]$.

Consider any $p, q \in K^* \backslash \mathscr{L}$, and choose representatives $P, Q \in \mathscr{L}$, respectively, and an R-basis b, b' of P such that $t^m b$, $t^n b'$ is an R-basis of Q for some $m, n \in \mathbb{Z}$. Now the sequences $Rt^i b + Rb'$, $i = 0, \ldots, m$, $Rt^m b + Rt^j b'$, $j = 0, \ldots, n$ in $\mathscr{L}$ determine a path in T connecting p to p'. It follows that T is connected.

If $p, q \in K^* \backslash \mathscr{L}$ and $d(p, q) = 1$ then we can choose representatives $P, Q \in \mathscr{L}$, respectively, and an R-basis b, b' of P such that $t^m b, t^{m+1} b'$ is an R-basis of Q for some integer m. Replacing Q by $t^{-m} Q$ we may assume that Q is a maximal submodule of P and tP is a maximal submodule of Q.

If T is not a tree then there exists a sequence $p_0, p_1, \ldots, p_n = p_0$ in $K^*\backslash\mathscr{L}$ such that $p_{i-1} \neq p_{i+1}$ for all $i \in [1, n-1]$, and $d(p_{i-1}, p_i) = 1$ for all $i \in [1, n]$. By the preceding paragraph we can construct a sequence $P_0, P_1, \ldots, P_n = t^m P_0$ in $\mathscr{L}$ such that $tP_{i-1} \neq P_{i+1}$ for all $i \in [1, n-1]$, and for each $i \in [1, n]$, P_i is a maximal submodule of P_{i-1} and tP_{i-1} is a maximal submodule of P_i. Hence, for each $i \in [1, n-1]$, tP_{i-1} and P_{i+1} are distinct maximal submodules of P_i, so $tP_{i-1} + P_{i+1} = P_i$. By reverse induction $tP_{i-1} + P_n = P_i$, and in particular $tP_0 + P_n = P_1$. Now $P_0 \supset P_n = t^m P_0$, so $m \geqslant 1$ and $tP_0 \supseteq P_n$, which means that $tP_0 + P_n = tP_0$. Hence $P_1 = tP_0 + P_n = tP_0 \subset P_1$, a contradiction. Thus T is a G-tree.

Let

$$p = \left[R\begin{pmatrix}1\\0\end{pmatrix} + R\begin{pmatrix}0\\1\end{pmatrix}\right], \quad p' = \left[R\begin{pmatrix}t\\0\end{pmatrix} + R\begin{pmatrix}0\\1\end{pmatrix}\right]$$

in $K^*\backslash\mathscr{L}$, and $e = (p, p')$ in E. Then $\iota e = p, \tau e = \{p, p'\}$.

We claim that the subgraph $Y = \{\iota e, e, \tau e\}$ is a G-transversal in T. It is clear that $\iota e, \tau e$ are in different G-orbits, so it suffices to show $E = Ge$. Now $K^*\backslash\mathscr{L} = Gp$ so each edge of T lies in the G-orbit of an edge of the form (p, q), where $d(p, q) = 1$. Here

$$q = \left[R\begin{pmatrix}a\\b\end{pmatrix} + R\begin{pmatrix}c\\d\end{pmatrix}\right]$$

for some $a, b, c, d \in R$ such that $ad - bc = t$. Hence, for a unique r modulo tR,

$$q = \left[R\begin{pmatrix}t\\0\end{pmatrix} + R\begin{pmatrix}r\\1\end{pmatrix}\right] = \begin{pmatrix}1 & r\\0 & 1\end{pmatrix}p'$$

or

$$q = \left[R\begin{pmatrix}0\\t\end{pmatrix} + R\begin{pmatrix}1\\r\end{pmatrix}\right] = \begin{pmatrix}0 & 1\\1 & r\end{pmatrix}p'.$$

Thus $E = Ge$, as desired. We remark that the neighbours of ιe are indexed by two copies of k.

By the Structure Theorem 4.1, $\mathrm{GL}_2(K) = G_{\iota e} \underset{G_e}{*} G_{\tau e}$. Notice that $G_{\iota e} = G_p = \mathrm{GL}_2(R) \cdot K^*$. Also, $p' = gp$, where

$$g = \begin{pmatrix}0 & t\\1 & 0\end{pmatrix},$$

so $G_{p'} = gG_p g^{-1}$, and it follows that

$$G_e = G_{p,p'} = \begin{pmatrix}U & tR\\R & U\end{pmatrix} \cdot K^*.$$

Finally, $G_{\tau e} = G_{\{p,p'\}}$ has index 2 in G_e and contains g which is not an element of $\mathrm{GL}_2(R)$, so

$$G_{\tau e} = \begin{pmatrix} U & tR \\ R & U \end{pmatrix} \cdot K^* \cup \begin{pmatrix} tR & tU \\ U & tR \end{pmatrix} \cdot K^*.$$

In summary, $\mathrm{GL}_2(K) = A \underset{C}{*} B$, where

$$A = \mathrm{GL}_2(R) \cdot K^*,$$

$$B = \begin{pmatrix} U & tR \\ R & U \end{pmatrix} \cdot K^* \cup \begin{pmatrix} R & U \\ t^{-1}U & R \end{pmatrix} \cdot K^*,$$

$$C = \begin{pmatrix} U & tR \\ R & U \end{pmatrix} \cdot K^*.$$

(ii) Let $H = \mathrm{SL}_2(K)$ and view T as an H-tree. By considering determinants it is not difficult to show that $K^* \backslash \mathscr{L}$ has exactly two H-orbits, Hp, Hgp, where

$$g = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}.$$

The above argument then shows that E has two H-orbits He, Hge, and one can compute that $\mathrm{SL}_2(K) = A \underset{C}{*} B$, where

$$A = \mathrm{SL}_2(R),$$

$$B = g\,\mathrm{SL}_2(R) g^{-1} = \left\{ \begin{pmatrix} a & tc \\ b/t & d \end{pmatrix} \middle| \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(R) \right\}$$

and

$$C = A \cap B = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(R) \mid c \in tR \right\}.$$

In fact we can contract all edges in Hge and obtain a new H-tree $\bar{T}$ with vertex set $K^* \backslash \mathscr{L}$ and edge set $\{\{v,w\} \mid v, w \in K^* \backslash \mathscr{L}$ with $d(v,w) = 1\}$. Here $\{v,w\}$ has v, w as its vertices, with the initial vertex being the one lying in Hp. One can think of T as being the barycentric subdivision of $\bar{T}$.

(iii) Consider the case where $K = k(x)$ the field of rational functions in an indeterminate x, $R = \{f/g \mid f, g \in k[x], \deg f \leqslant \deg g\} = k[x^{-1}]_{(x^{-1})}$ and the uniformizer is $t = x^{-1}$. The above $\mathrm{GL}_2(K)$-tree T is then a $\mathrm{GL}_2(k[x])$-tree, and one can verify that

$$\left[R\begin{pmatrix} 1 \\ 0 \end{pmatrix} + R\begin{pmatrix} 0 \\ t^n \end{pmatrix} \right], \quad n \geqslant 0,$$

is a $\mathrm{GL}_2(k[x])$-transversal in $K^* \backslash \mathscr{L}$. The Structure Theorem 4.1 then shows that $\mathrm{GL}_2(k[x]) = \mathrm{GL}_2(k) \underset{T_2(k)}{*} T_2(k[x])$, where T_2 denotes lower triangular 2×2 matrices. ∎

5.4 Examples. Let k be a field and x, y indeterminates. Write $k\langle x, y\rangle$ for the free associative k-algebra in x, y, so x, y commute with the elements of k but not with each other. Write $k[x, y]$ for the polynomial ring over k in x, y, the abelianization of $k\langle x, y\rangle$.

Let $R = k\langle x, y\rangle$ or $k[x, y]$, and let G be the group of all k-algebra automorphisms of R, that is, ring automorphisms of R which act as the identity on k. Such an automorphism ϕ is completely specified by the ordered pair $(\phi x, \phi y)$, so we can use ordered pairs to describe automorphisms.

Let X be the graph whose vertices are the k-subspaces of R and whose edges are the inclusion relations between them. Thus an edge corresponds to an inclusion $v \subseteq w$, where v, w are k-subspaces of R, and we specify that v, w are the initial and terminal vertex, respectively. Clearly X is a G-graph.

Consider the vertices $v = k + kx, w = k + kx + ky$ of X and let e be the edge of X joining them. Let Y be the subgraph $\{v, e, w\}$ of X, and let $T = GY$. Clearly T is a G-subgraph of X. By some rather involved manipulations with leading forms one can show that T is a G-tree; see Cohn (1985). Since T clearly has fundamental G-transversal Y, it follows that $G = A \underset{C}{*} B$, where

$$A = G_v = \{(\lambda + \alpha x, \delta y + f(x)) \mid f(x) \in k[x], \alpha, \delta, \lambda \in k, \alpha\delta \neq 0\},$$

the group of *x-based de Jonquières automorphisms of* R,

$$B = G_w = \{(\lambda + \alpha x + \beta y, \mu + \gamma x + \delta y) \mid \alpha, \beta, \gamma, \delta, \lambda, \mu \in k, \alpha\delta \neq \beta\gamma\},$$

the group of *affine automorphisms* of R, and

$$C = G_e = \{(\lambda + \alpha x, \mu + \gamma x + \delta y) \mid \alpha, \gamma, \delta, \lambda, \mu \in k, \alpha\delta \neq 0\},$$

the group of affine x-based de Jonquières automorphisms of R.

In particular, G is the same for both $R = k\langle x, y\rangle$ and $R = k[x, y]$. ∎

6 The exact sequence for a tree

6.1 Definitions. By a *G-module* M we mean an abelian group M which is a G-set such that G acts by abelian group automorphisms on M. An additive G-map between G-modules is called a *G-linear* map.

For any G-set V we write $\mathbb{Z}V$ or $\mathbb{Z}[V]$ for the free abelian group on V, and write the elements as formal sums $\sum\limits_{v \in V} n_v v$, with $n_v \in \mathbb{Z}$ being

0 for all but finitely many $v \in V$. This is a G-module with G-action $g\left(\sum_{v \in V} n_v v\right) = \sum_{v \in V} n_v gv$. The G-linear map $\varepsilon: \mathbb{Z}V \to \mathbb{Z}$ with $\varepsilon\left(\sum_{v \in V} n_v v\right) = \sum_{v \in V} n_v$ is called the *augmentation map of* V. The kernel of ε is a G-submodule $\omega\mathbb{Z}V$ of $\mathbb{Z}V$ called the *augmentation module* of V. ∎

6.2 Definitions. Let (X, V, E, ι, τ) be a G-graph.

By the *boundary map of* X we mean the G-linear map $\partial: \mathbb{Z}E \to \mathbb{Z}V$ with $\partial(e) = \tau e - \iota e$ for all $e \in E$, that is, $\partial\left(\sum_{e \in E} n_e e\right) = \sum_{e \in E} n_e \tau e - \sum_{e \in E} n_e \iota e$. The sequence

(1) $$0 \to \mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is then a *complex*, that is the composite of any two consecutive maps is zero; it is called the *augmented cellular chain complex of* X, or simply the *complex for* X.

The abelian groups in the complex are called the *terms* of the complex. A complex is said to be *exact* at a term if the kernel of the map outward from that term is precisely the image of the map into that term. The complex itself is *exact* if it is exact at each term. ∎

Notice that the complex (1) is exact at $\mathbb{Z}$, that is, ε is surjective, because V is nonempty.

We now examine exactness at each of the terms.

6.3 Lemma. *X is connected if and only if $\mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is exact.*

Proof. The cokernel of ∂ can be expressed in the form $\mathbb{Z}C$, where C is the G-set obtained from V by identifying $\tau e = \iota e$ for all $e \in E$. Since C can be viewed as the set of components of X, the result is proved. ∎

6.4 Lemma. *X is a forest if and only if $0 \to \mathbb{Z}V \xrightarrow{\partial} \mathbb{Z}E$ is exact.*

Proof. If X is not a forest then there is a simple closed path $v_0, e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}, v_n = v_0$ in X, so $\partial(\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n) = (-v_0 + v_1) + (-v_1 + v_2) + \cdots + (-v_{n-1} + v_n) = -v_0 + v_n = 0$. But $\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n \neq 0$ so ∂ is not injective.

The converse is an easy exercise, and in fact we shall now see how to construct a one-sided inverse for ∂. ∎

6.5 Definitions. Let $T = (T, V, E, \iota, \tau)$ be a G-tree.

For any vertices v, w of T there is a geodesic $v = v_0, e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}, v_n = w$, and we write $T[v, w]$ for the element $\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n$ of $\mathbb{Z}E$. Notice that $T[w, v] = -T[v, w]$ and $T[v, w] = T[v, u] + T[u, w]$ for all vertices u. Since $\mathbb{Z}V$ is free abelian we can extend the map $T[v, \text{-}]: V \to \mathbb{Z}E$, $w \mapsto T[v, w]$, to a map $\mathbb{Z}V \to \mathbb{Z}E$, and we denote this new map also by $T[v, \text{-}]$. ∎

6.6 Theorem. *If T is a G-tree then there is an exact sequence of G-modules $0 \to \mathbb{Z}ET \xrightarrow{\partial} \mathbb{Z}VT \xrightarrow{\varepsilon} \mathbb{Z} \to 0$; hence $\omega\mathbb{Z}VT \approx \mathbb{Z}ET$ as G-modules.*

Proof. Let v be a vertex of T. For any edge e of T, $T[v, \partial e] = -T[v, \iota e] + T[v, \tau e] = T[\iota e, v] + T[v, \tau e] = T[\iota e, \tau e] = e$. Thus the composite $\mathbb{Z}ET \xrightarrow{\partial} \mathbb{Z}VT \xrightarrow{T[v,\text{-}]} \mathbb{Z}ET$ is the identity, so ∂ is injective. ∎

In Theorem 9.2 we shall see the corresponding exact sequence for a connected G-graph.

In proving that the standard graph is a tree in the next section, we shall use the appropriate part of Lemma 6.4. However, the above proof of Theorem 6.6 provides the actual motivation for the argument. It will be useful to have the following concepts.

6.7 Definitions. Let M be a group on which G acts by group automorphisms. Let us write M additively, although it need not be abelian.

By the *semidirect product* $G \ltimes M$ or $M \rtimes G$ we mean the group with underlying set $G \times M$ and multiplication $(g, m)(g', m') = (gg', m + gm')$, suggestive of 2×2 matrices

$$\begin{pmatrix} G & M \\ 0 & 1 \end{pmatrix}.$$

Now let M be abelian, so M is a G-module.

By a *derivation* $d: G \to M$ we mean a function such that $d(xy) = d(x) + xd(y)$ for all $x, y \in G$.

It is straightforward to verify that a function $d: G \to M$ is a derivation if and only if the function $(1, d): G \to G \ltimes M, g \mapsto (g, dg)$, is a group homomorphism.

For any $m \in M$, the function $\operatorname{ad} m: G \to M, g \mapsto gm - m$, is a derivation; in $G \ltimes M$ it corresponds to right conjugation by $(1, m)$.

A derivation $d: G \to M$ is *inner* if $d = \operatorname{ad} m$ for some $m \in M$; otherwise d is *outer*. ∎

6.8 Example. Let T be a G-tree and v a vertex of T.

The inner derivation $\operatorname{ad} v: G \to \mathbb{Z}VT$ has image lying in $\omega\mathbb{Z}VT \approx \mathbb{Z}ET$. The resulting derivation $G \to \mathbb{Z}ET$ is given by $g \mapsto T[v, gv]$, and is denoted $T[v, \ v]$. ∎

7 The fundamental group and its tree

We now analyze the fundamental group of a graph of groups and verify that the standard graph is a tree. Throughout this section we fix the following terminology introduced in Section 3:

7.1 Notation. Let $(G(-), Y)$ be a graph of groups with connected graph (Y, V, E, ι, τ), vertex groups $G(v), v \in V$, edge groups $G(e) \subseteq G(\iota e), e \in E$, and edge functions $t_e: G(e) \to G(\tau e), g \mapsto g^{t_e}, e \in E$.

Let Y_0 be a maximal subtree of Y and v_0 a vertex of Y.

Let $G = \pi(G(-), Y, Y_0)$, the group presented with

generating set: $\{t_e \mid e \in E\} \vee \bigvee_{v \in V} G(v)$

relations: the relations for $G(v), v \in V$;

$$gt_e = t_e g^{t_e} \quad \text{for all } e \in E, \quad g \in G(e) \subseteq G(\iota e) \quad \text{so } g^{t_e} \in G(\tau e);$$

$$t_e = 1 \quad \text{for all } e \in EY_0.$$

Let $T = T(G(-), Y, Y_0)$, the G-graph presented with generating set Y, relations saying y is $G(y)$-stable, and incidence functions given by $\iota(ge) = g\iota e, \tau(ge) = gt_e\tau e$ for all $g \in G, e \in E$. ∎

7.2 Lemma. *Suppose that H is a group and that there are specified group homomorphisms $\alpha_v: G(v) \to H, v \in VY$, and a function $\alpha: E \to H$, such that $\alpha_{\iota e}(g)\alpha(e) = \alpha(e)\alpha_{\tau e}(g^{t_e})$ for all $e \in E$, $g \in G(e) \subseteq G(\iota e)$. For $v, w \in VY$, define $\alpha(v, w)$ to be the element $\alpha(e_1)^{\varepsilon_1} \cdots \alpha(e_n)^{\varepsilon_n}$ of H where $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ is the Y_0-geodesic from v to w.*

Then there exists a group homomorphism $\beta: G \to H$ defined on the given generating set as

$$\beta(g) = \alpha(v_0, v)\alpha_v(g)\alpha(v, v_0) \quad \textit{for all } g \in G(v), \quad v \in V,$$

and

$$\beta(t_e) = \alpha(v_0, \iota e)\alpha(e)\alpha(\tau e, v_0) \quad \textit{for all } e \in E.$$

Proof. We need check only that β respects the relations of G.

It is easy to show that, for any $u, v, w \in V$, $\alpha(u, v)\alpha(v, w) = \alpha(u, w)$, and $\alpha(w, v) = \alpha(v, w)^{-1}$.

For $v \in V$, consider the restriction of β to the subset $G(v)$ of the generating set. It is obtained by composing α_v with left conjugation by $\alpha(v_0, v) = \alpha(v, v_0)^{-1}$, so is a group homomorphism, which means that the relations of $G(v)$ are respected.

For each $e \in E$ and $g \in G(e) \subseteq G(\iota e)$,

$$\begin{aligned}\beta(g)\beta(t_e) &= \alpha(v_0, \iota e)\alpha_{\iota e}(g)\alpha(\iota e, v_0)\alpha(v_0, \iota e)\alpha(e)\alpha(\tau e, v_0)\\ &= \alpha(v_0, \iota e)\alpha_{\iota e}(g)\alpha(e)\alpha(\tau e, v_0) = \alpha(v_0, \iota e)\alpha(e)\alpha_{\tau e}(g^{t_e})\alpha(\tau e, v_0)\\ &= \alpha(v_0, \iota e)\alpha(e)\alpha(\tau e, v_0)\alpha(v_0, \tau e)\alpha_{\tau e}(g^{t_e})\alpha(\tau e, v_0)\\ &= \beta(t_e)\beta(g^{t_e}).\end{aligned}$$

For each $e \in EY_0, \beta(t_e) = \alpha(v_0, \iota e)\alpha(e)\alpha(\tau e, v_0) = \alpha(v_0, \iota e)\alpha(\iota e, \tau e)\alpha(\tau e, v_0) = \alpha(v_0, v_0) = 1$.

Thus β does respect the relations of G and we have the desired group homomorphism $\beta: G \to H$. ∎

7.3 Definition. Let P be the group presented with

generating set: $\{u_e \mid e \in E\} \vee \bigvee_{v \in V} G(v)$

relations: the relations for $G(v), v \in V$;

$gu_e = u_e g^{t_e}$ for all $e \in E$, $g \in G(e) \subseteq G(\iota e)$, so $g^{t_e} \in G(\tau e)$.

The *fundamental group of* $(G(-), Y)$ *with respect to* v_0, denoted $\pi(G(-), Y, v_0)$, is defined to be the subgroup of P consisting of all elements p for which there exists an expression $p = g_0 u_{e_1}^{\varepsilon_1} u_1 \cdots g_{n-1} u_{e_n}^{\varepsilon_n} g_n$ and a closed path $v_0, e_1^{\varepsilon_1}, v_1, \ldots, e_n^{\varepsilon_n}, v_n = v_0$ in Y at v_0, with $g_i \in G(v_i)$ for all $i \in [0, n]$.

There is a well-defined group homomorphism $P \to G$ sending each u_e to t_e. Using the function $E \to P, e \mapsto u_e$, we can apply Lemma 7.2 to get a group homomorphism $\beta: G \to P$, and the composite $G \xrightarrow{\beta} P \to G$ is then the identity, so β is injective. It is not difficult to show that $\beta G = \pi(G(-), Y, v_0)$, and we have $\pi(G(-), Y, Y_0) = G \approx \beta G = \pi(G(-), Y, v_0)$.

Hence the isomorphism class of $\pi(G(-), Y, Y_0)$ is independent of the choice of Y_0. Where we are dealing with abstract group properties we shall sometimes speak of the *fundamental group* of $(G(-), Y)$ and write $\pi(G(-), Y)$.

Here P is a semidirect product of G by the normal closure of $\{u_e \mid e \in EY_0\}$. ∎

7.4 Theorem. *If U is a nonempty set such that $|U|$ is uniquely divisible by $|G(v)|$ for each $v \in V$, then there exists a group homomorphism $G \to \mathrm{Sym}\, U$ such that the composite $G(v) \to G \to \mathrm{Sym}\, U$ is injective for each $v \in V$.*

Proof. For each $v \in V$, $|G(v)|$ divides $|U|$ so we can partition U into copies of $G(v)$, and hence define a free $G(v)$-action on U; we denote by $\alpha_v: G(v) \to \mathrm{Sym}\, U$ the resulting injective homomorphism.

Consider any $e \in E$. The free $G(\bar{\iota}e)$ and $G(\bar{\tau}e)$ actions on U induce free $G(e)$-actions on U via the maps $G(e) \subseteq G(\bar{\iota}e)$ and $t_e: G(e) \to G(\bar{\tau}e)$, respectively; let us denote the corresponding $G(e)$-sets as U_ι, U_τ. These two free $G(e)$-sets are isomorphic because $|U|$ is uniquely divisible by $|G(e)|$, so there exists a $G(e)$-isomorphism $\alpha(e): U_\tau \to U_\iota$. Thus for all $g \in G(e)$, $u \in U$, we have $\alpha(e)(\alpha_{\tau e}(g^{t_e})u) = \alpha_{\iota e}(g)(\alpha(e)(u))$. Hence $\alpha(e)$ is an element of $\mathrm{Sym}\, U$ such that for all $g \in G(e)$, $\alpha(e)\alpha_{\tau e}(g^{t_e}) = \alpha_{\iota e}(g)\alpha(e)$.

By Lemma 7.2, there is a group homomorphism $\beta: G \to \mathrm{Sym}\, U$ such that for each $v \in VY$, the composite $G(v) \to G \to \mathrm{Sym}\, U$ is conjugate to α_v, so is injective, since α_v is injective. ∎

7.5 Corollary. *$G(v) \to G$ is injective for each $v \in VY$.*

Proof. Let U be any infinite set such that $|U| > |G(v)|$ for all $v \in VY$. Then $|U|$ is uniquely divisible by $|G(v)|$ for all $v \in VY$. It now follows from Theorem 7.4 that for each $v \in VY$, $G(v) \to G$ is injective. ∎

Henceforth the $G(y)$, $y \in Y$, will be treated as subgroups of G.

We now come to the main result of the chapter which will have many applications in the sequel.

7.6 Theorem. *T is a tree.*

Proof. To see that T is connected we consider the set CT of components of T. There is a natural map $VT \to CT$, $v \mapsto [v]$. This is a G-map, and $[\iota e] = [\tau e]$ for all $e \in ET$. Recall that VT is the G-set generated by V with relations saying that each $v \in V$ is $G(v)$-stable. Hence CT is the G-set generated by V with relations saying that each $[v] \in CT$ is $G(v)$-stable, and additional relations saying that for each $e \in E$, $[\iota e] = [\tau e]$, that is, $[\bar{\iota}e] = [t_e \bar{\tau}e] = t_e[\bar{\tau}e]$. As e ranges over EY_0 the latter relations have the form $[\bar{\iota}e] = [\bar{\tau}e]$ for all $e \in EY_0$, and thus $[v] = [v_0]$ for all $v \in V$, so CT

has only one G-orbit. Now $[v_0]$ is $G(v)$-stable for each $v\in V$, since $[v]=[v_0]$; and $[v_0]$ is t_e-stable for each $e\in EY$, since $[v_0]=[\iota e]=t_e[\tau e]=t_e[v_0]$. Thus $[v_0]$ is G-stable, so CT consists of a single element. Hence T is connected.

By Lemma 6.4 it now sufficies to show that the boundary map $\partial:\mathbb{Z}ET\to\mathbb{Z}VT$ is injective.

Without knowing that T is a tree, we will be able to construct the derivation $T[v_0,\_v_0]:G\to\mathbb{Z}ET$ using the generators and relations of G.

For each $v\in V$, there is an obvious group homomorphism $\alpha_v:G(v)\to G\ltimes\mathbb{Z}ET$, $g\mapsto(g,0)$. Define a function $\alpha:E\to G\ltimes\mathbb{Z}ET$, $e\mapsto(t_e,e)$.

If $g\in G(e)$, $e\in E$, then $\alpha_{\iota e}(g)\alpha(e)=(g,0)(t_e,e)=(gt_e,ge)=(t_eg^{t_e},e)=(t_e,e)(g^{t_e},0)=\alpha(e)\alpha_{\tau e}(g^{t_e})$, and the hypotheses of Lemma 7.2 are satisfied.

As in Lemma 7.2, for $v,w\in V$, if $e_1^{\varepsilon_1},\dots,e_n^{\varepsilon_n}$ is the Y_0-geodesic from v to w, then $\alpha(v,w)=\alpha(e_1)^{\varepsilon_1}\cdots\alpha(e_n)^{\varepsilon_n}=(1,e_1)^{\varepsilon_1}\cdots(1,e_n)^{\varepsilon_n}=(1,Y_0[v,w])$, where $Y_0[v,w]=\varepsilon_1e_1+\cdots+\varepsilon_ne_n\in\mathbb{Z}EY_0\subseteq\mathbb{Z}ET$. By Lemma 7.2 there exists a group homomorphism $\beta:G\to G\ltimes\mathbb{Z}ET$ such that

$$\begin{aligned}
&\text{if } g\in G(v),\quad v\in V\quad \text{then } \beta(g)=\alpha(v_0,v)\alpha_v(g)\alpha(v,v_0)\\
&\quad=(1,Y_0[v_0,v])(g,0)(1,Y_0[v,v_0])=(g,Y_0[v_0,v]+gY_0[v,v_0]);\\
&\text{if } e\in E\quad \text{then } \beta(t_e)=\alpha(v_0,\iota e)\alpha(e)\alpha(\tau e,v_0)\\
&\quad=(1,Y_0[v_0,\iota e])(t_e,e)(1,Y_0[\tau,v_0])\\
&\quad=(t_e,Y_0[v_0,\iota e]+e+t_eY_0[\tau e,v_0]).
\end{aligned}$$

It is clear that β has the form $(1,d):G\to G\ltimes\mathbb{Z}ET$, so we have a map $G\to\mathbb{Z}ET$, denoted $T[v_0,\_v_0]$, such that

(1) $T[v_0,\_v_0]$ is a derivation;

(2) if $g\in G(v), v\in V$, then $T[v_0,gv_0]=Y_0[v_0,v]+gY_0[v,v_0]$;

(3) if $e\in E$ then $T[v_0,t_ev_0]=Y_0[v_0,\iota e]+e+t_eY_0[\tau e,v_0]$.

We claim there is a well-defined additive map $T[v_0,\text{-}]:\mathbb{Z}VT\to\mathbb{Z}ET$ such that

(4) $T[v_0,gv]=T[v_0,gv_0]+gY_0[v_0,v]$ for all $g\in G$, $v\in V$.

Thus suppose $gv=g'v'$ with $g,g'\in G$, $v,v'\in V$. Then $v=v'$ and $g'=gh$ for some $h\in G(v)$ so

$$\begin{aligned}
&T[v_0,g'v_0]+g'Y_0[v_0,v]\\
&\quad=T[v_0,ghv_0]+ghY_0[v_0,v]\\
&\quad=\{T[v_0,gv_0]+gT[v_0,hv_0]\}+ghY_0[v_0,v]\ \text{by (1)}\\
&\quad=T[v_0,gv_0]+g\{Y_0[v_0,v]+hY_0[v,v_0]\}+ghY_0[v_0,v]\ \text{by (2)}\\
&\quad=T[v_0,gv_0]+gY_0[v_0,v].
\end{aligned}$$

Thus $T[v_0,\text{-}]:\mathbb{Z}VT \to \mathbb{Z}ET$ is well-defined.

For any $g \in G$, $e \in E$ we have

$$\begin{aligned}
T[v_0, \partial ge] &= -T[v_0, g\iota e] + T[v_0, gt_e\tau e] \\
&= \{gY_0[\iota e, v_0] - T[v_0, gv_0]\} + \{T[v_0, gt_e v_0] + gt_e Y_0[v_0, \tau e]\} \\
&\quad \text{by (4)} \\
&= gY_0[\iota e, v_0] - T[v_0, gv_0] + \{T[v_0, gv_0] + gT[v_0, t_e v_0]\} \\
&\quad + gt_e Y_0[v_0, \tau e] \text{ by (1)} \\
&= gY_0[\iota e, v_0] + gT[v_0, t_e v_0] + gt_e Y_0[v_0, \tau e] \\
&= gY_0[\iota e, v_0] + g\{Y_0[v_0, \iota e] + e + t_e Y_0[\tau e, v_0]\} \\
&\quad + gt_e Y_0[v_0, \tau e] \text{ by (3)} \\
&= ge.
\end{aligned}$$

It follows that $\mathbb{Z}ET \xrightarrow{\partial} \mathbb{Z}VT \xrightarrow{T[v_0,\text{-}]} \mathbb{Z}ET$ is the identity map, so ∂ is injective, as desired. ∎

If H is any subgroup of G then T is an H-tree so H is isomorphic to the fundamental group of the graph of groups associated with T with respect to a fundamental H-transversal and connecting elements. If H acts freely on the edges the structure of H is fairly easy to describe.

7.7 Theorem. *If H is a subgroup of G which intersects each G-conjugate of each edge group $G(e)$ trivially then $H = F * \underset{i \in I}{*} H_i$ for some free subgroup F, and subgroups H_i of the form $H \cap gG(v)g^{-1}$ as g ranges over a certain set of double coset representatives in $H \backslash G/G(v)$ and v ranges over VY.* ∎

7.8 The Kurosh Subgroup Theorem. *If H is a subgroup of a free product $\underset{v \in V}{*} G(v)$ then $H = F * \underset{i \in I}{*} H_i$ for some free subgroup F, and subgroups H_i of the form $H \cap gG(v)g^{-1}$ as g ranges over a certain set of double coset representatives in $H \backslash G/G(v)$ and v ranges over V.* ∎

There are even better actions such as those occurring in Propositions 4.5, 4.6 and Corollary 4.9.

7.9 Proposition. *If a subgroup H of G intersects each G-conjugate of each vertex group trivially then H is free.*

For example, if H is torsion-free and the vertex groups are torsion groups then H is free. ∎

7.10 Proposition. *If $G \to A$ is a homomorphism of groups which is injective on each vertex group then the kernel is free.* ∎

7.11 Proposition. *Every finite subgroup of G lies in some conjugate of some vertex group.* ∎

8 Free groups

8.1 Definitions. Let Y be a connected graph, Y_0 a maximal subtree of Y, and v_0 a vertex of Y.

Form the graph of groups $(G(-), Y)$ with $G(y) = 1$ for all $y \in Y$.

In Example 3.5 (i), we defined the *fundamental group of Y with respect to Y_0* to be $\pi(Y, Y_0) = \pi(G(-), Y, Y_0)$; this is essentially the free group on $EY - EY_0$.

We define the *fundamental group of Y with respect to v_0* as $\pi(Y, v_0) = \pi(G(-), Y, v_0)$; that is, the subgroup of the free group on EY consisting of all elements $e_1^{\varepsilon_1} e_2^{\varepsilon_2} \cdots e_n^{\varepsilon_n}$, where $e_1^{\varepsilon_1}, e_2^{\varepsilon_2}, \ldots, e_n^{\varepsilon_n}$ is a closed path in Y at v_0. This agrees with the usual notion of the fundamental group of Y at v_0 consisting of homotopy classes of closed paths at v_0.

In Definition 7.3 it was shown that $\pi(Y, Y_0) \approx \pi(Y, v_0)$.

Since the isomorphism type is independent of all choices we agreed to speak of the *fundamental group* of Y and write $\pi(Y)$, thinking of it as the free group of rank $|EY - EY_0|$.

Let $G = \pi(Y, Y_0)$ and write $T = T(G(-), Y, Y_0)$. We treat T as having a distinguished vertex v_0. By Theorem 7.6, T is a tree, and from the construction we see T is a G-free G-tree with $G \backslash T \approx Y$.

Hence the corresponding map $T \to Y$ is an isomorphism on the neighbourhood of each vertex. Any tree with the latter property is called the *universal covering tree* of Y, which agrees with the usual notion of the universal covering space of Y. It is not difficult to show that universal covering trees for Y are unique up to unique isomorphism, as trees with distinguished vertex.

For example, suppose Y has only one vertex, so consists of loops. Here G is free on EY, and T is the Cayley graph of G with respect to EY. The free groups of ranks 1 and 2 were illustrated in Example 2.2(ii), (iv).

We define rank $Y = |EY - EY_0|$, so rank G = rank Y.

For a free group F of finite rank, we define the *Euler characteristic* of F to be $\chi(F) = 1 - \text{rank}\, F$.

For a finite graph Y the *Euler characteristic of* Y is

$$\chi(Y) = |VY| - |EY| = |VY_0| - |EY| = (1 + |EY_0|) - |EY|$$
$$= 1 - |EY - EY_0| = 1 - \operatorname{rank} Y = 1 - \operatorname{rank} G = \chi(G). \quad \blacksquare$$

Let us repeat one of the above observations and then combine it with Corollary 4.2.

8.2 Theorem. *G is freely generated by a subset S if and only if the Cayley graph $X(G, S)$ is a G-tree.* $\blacksquare$

8.3 Theorem. *There exists a G-free G-tree if and only if G is a free group.* $\blacksquare$

The former property is clearly inherited by subgroups.

8.4 The Nielsen–Schreier Theorem. *Every subgroup of a free group is free.* $\blacksquare$

A closer analysis enables us to describe the ranks of the subgroups.

8.5 The Schreier Index Formula. *If G is a free group of finite rank r and H a subgroup of G of finite index n then H is a free group of rank $1 + n(r - 1)$. In terms of Euler characteristics, $\chi(H) = (G:H)\chi(G)$.*

Proof. Let S be a free generating set of G, and let $T = X(G, S)$ so T is H-free and G-free. By Corollary 4.2, $G \approx \pi(G\backslash T)$, $H \approx \pi(H\backslash T)$. Here $G\backslash T$ is a finite graph, so $\chi(G) = \chi(G\backslash T)$. As ET is G-free, it is $|G\backslash ET|$ copies of G, and hence $(G:H)|G\backslash ET|$ copies of H; thus $(G:H)|G\backslash ET| = |H\backslash ET|$. The analogous result holds for VT, so $H\backslash T$ is a finite graph and $\chi(H\backslash T) = (G:H)\chi(G\backslash T)$. Hence, $\chi(H) = \chi(H\backslash T) = (G:H)\chi(G\backslash T) = (G:H)\chi(G)$. $\blacksquare$

There are many other results about free groups which can be proved using trees and we conclude this section with a sampling.

8.6 Theorem. *If G is free of finite rank and ϕ an automorphism of G then the subgroup H of elements of G stabilized by ϕ is free of finite rank.*

Proof. Let T be the Cayley graph of G with respect to a free generating set S of G. By Theorem 8.2, T is a G-free G-tree.

Since $VT = G$ we have an H-map $\phi: VT \to VT$, and hence an H-subset

E' of ET consisting of the edges e such that the T-geodesic from $\phi(\iota e)$ to $\phi(\tau e)$ contains e.

We shall show $H\backslash E'$ is finite. Consider any $e \in E'$, so $\iota e = g$, $\tau e = gs$ for some $g \in G$, $s \in S$, and the T-geodesic from $\phi(\iota e) = \phi(g)$ to $\phi(\tau e) = \phi(gs) = \phi(g)\phi(s)$ contains $\iota e = g$. Applying $\phi(g)^{-1}$ we see that the T-geodesic from 1 to $\phi(s)$ contains $\phi(g)^{-1}g$. It is easy to check that the map $H\backslash G \to G$, $Hg \mapsto \phi(g)^{-1}g$, is well-defined and injective. Since S is finite, there are only finitely many pairs (Hg, s) such that $\phi(g)^{-1}g$ lies in the T-geodesic from 1 to $\phi(s)$. Hence, $H\backslash E'$ is finite.

We now ignore the G-action, and view T solely as H-free H-tree.

If $e \in ET - E'$, then $\phi(\iota e)$, $\phi(\tau e)$ lie in the same component of $T - \{e\}$. Hence we can reorient T so that for each $e \in ET - E'$, τe lies in the same component of $T - \{e\}$ as $\phi(\iota e)$, $\phi(\tau e)$. Notice this reorientation respects the H-action.

Also, e is the first edge in the geodesic from ιe to $\phi(\iota e)$. Hence, for any vertex v of T, any edge in $ET - E'$ having v as initial vertex must be the first edge in the T-geodesic from v to $\phi(v)$, but there is at most one such edge. Thus in $T - E'$, each vertex is the initial vertex of at most one edge.

By Corollary 4.2, $H \approx \pi(H\backslash T)$, and it suffices to show that $H\backslash T$ has finite rank. Since $H\backslash E'$ is finite, the graph $(H\backslash T) - (H\backslash E')$ has finitely many components, and it suffices to show that each component has finite rank. But $(H\backslash T) - (H\backslash E') = H\backslash(T - E')$, and here each vertex is the initial vertex of at most one edge. Hence each reduced closed path is oriented cyclically with attached paths pointing in. It follows that no two simple closed paths can ever be attached, and thus each component has rank at most one. ∎

This proves that H is finitely generated.

8.7 Conjecture. If G is free of rank n, and ϕ an automorphism of G, then the subgroup H of elements of G stabilized by ϕ is free of rank at most n.

Discussion. At the time of writing it is not known if rank H can be bounded by a function of n. ∎

8.8 Theorem. *If A and B are finitely generated subgroups of a free group G then $A \cap B$ is finitely generated.*

Proof. Write $C = A \cap B$. By Theorem 8.3 there exists a G-free G-tree T. Let v be a vertex of T, and let T_A, T_B, T_C be the subtrees of T generated by Av, Bv, Cv respectively, so closed under the actions of A, B, C respectively. Clearly T_C is contained in both T_A and T_B. Hence, there is a map $C\backslash T_C \to A\backslash T_A \times B\backslash T_B$, $Ct \mapsto (At, Bt)$, and it is easily seen to be injective.

But $A\backslash T_A$ and $B\backslash T_B$ are finite since A and B are finitely generated. Thus $C\backslash T_C$ is finite, so its fundamental group, C, is finitely generated. ∎

8.9 Conjecture. If A and B are nontrivial finitely generated subgroups of a free group then $\operatorname{rank}(A \cap B) \leqslant 2 - \operatorname{rank} A - \operatorname{rank} B + \operatorname{rank}(A)\operatorname{rank}(B)$, or equivalently $-\chi(A \cap B) \leqslant \chi(A)\chi(B)$.

Discussion. Howson (1954) showed that

$$-\chi(A \cap B) \leqslant 2\chi(A)\chi(B) - \chi(A) - \chi(B) + 2$$

and H. Neumann (1955) improved this to

$$-\chi(A \cap B) \leqslant 2\chi(A)\chi(B).$$

Burns (1969) improved this to

$$-\chi(A \cap B) \leqslant 2\chi(A)\chi(B) + \max\{\chi(A), \chi(B)\},$$

and the matter still rests there. See Nickolas (1985) for more details. ∎

Theorem 8.8 allows us to extend Theorem 8.6 to any finite set of automorphisms.

8.10 Corollary. *For any free group G of finite rank and finitely generated group A of automorphisms of G, the subgroup H of elements of G stabilized by A is free of finite rank.* ∎

9 Groups acting on connected graphs

This section shows that group actions on connected graphs arise from group actions on trees as in Proposition 4.6.

9.1 Notation. Throughout this section let X be a connected G-graph.

Choose a fundamental G-transversal Y in X with subtree Y_0, and denote the incidence functions by $\bar{\iota}, \bar{\tau}$. Choose a vertex v_0 in Y, and for each $e \in EY$ choose an element $t_e \in G$ such that $t_e \bar{\tau} e = \tau e$, with $t_e = 1$ if $e = EY_0$. Let $(G(-), Y)$ be the resulting graph of groups and write $P = \pi(G(-), Y, Y_0)$, $T = T(G(-), Y, Y_0)$. We treat v_0 as an element of Y_0, Y, X and T. ∎

9.2 Structure Theorem for groups acting on connected graphs. *There is a natural extension of groups $1 \to \pi(X) \to P \to G \to 1$.*

Further, $\pi(X)$ acts freely on T, and there is a natural isomorphism of G-graphs $\pi(X)\backslash T \approx X$. In particular, T is the universal covering tree of X.

The action of P on $\pi(X)$ by left conjugation induces a natural G-module structure on $\pi(X)^{ab}$, and there is an exact sequence of G-modules

(1) $$0 \to \pi(X)^{ab} \to \mathbb{Z}[EX] \xrightarrow{\partial} \mathbb{Z}[VX] \to \mathbb{Z} \to 0.$$

Proof. Let $v \in VY$. As in the proof of the Structure Theorem 4.1, the paths of length 1 in X starting at v are the sequences of the form $v, gt_e^{\frac{1}{2}(\varepsilon-1)}e^{\varepsilon}, gt_e^{\varepsilon}w$, where v, e^{ε}, w is a path in Y, and $g \in G_v$. Hence, as in the proof of Theorem 4.1, the homomorphism $P \to G$ is surjective. Let N be the kernel, so $G = P/N$.

For each $y \in Y$, the composite $G(y) \to P \to G$ is the natural embedding, so N does not meet any meet any vertex groups. Thus T is N-free and $N \approx \pi(N\backslash T)$ by Corollary 4.2.

The graph $N\backslash T$ is acted on by the group $P/N = G$; moreover, Y is a fundamental G-transversal, the $t_e, e \in EY$, are connecting elements, and the resulting graph of groups agrees with $(G(-), Y)$. As before, the paths of length 1 in $N\backslash T$ starting at v are the sequences of the form $v, gt_e^{\frac{1}{2}(\varepsilon-1)}e^{\varepsilon}, gt_e^{\varepsilon}w$, where v, e^{ε}, w is a path in Y, and $g \in G_v$. It is then not difficult to deduce that $N\backslash T \approx X$ as G-graphs, so $N \approx \pi(X)$. We shall treat this isomorphism as an identification, and we wish to make this precise.

For any element c of N, the path in T from v_0 to cv_0 maps to a closed path in X at v_0 which corresponds to the element of $\pi(X)$ which we identify with c.

We have P acting on $\pi(X)$ by left conjugation, so in the induced action on $\pi(X)^{ab}$, $\pi(X)$ acts trivially, and thus $\pi(X)^{ab}$ has the structure of a module over $P/\pi(X) = G$. The action under $g \in G$ sends an element of $\pi(X)^{ab}$ represented by a closed path q in X at v_0 to the element of $\pi(X)^{ab}$ represented by any *g-translate* of q, that is, a closed path p, gq, p^{-1} where p is a path in X from v_0 to gv_0. This action is independent of all choices.

The function which associates to a path $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ in X the element $\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n \in \mathbb{Z}[EX]$ induces a natural map $\pi(X)^{ab} \to \mathbb{Z}[EX]$ which is easily seen to be G-linear, and we have a complex (1).

Since X is connected, (1) is exact at $\mathbb{Z}[VX]$ by Lemma 6.3.

Choose a maximal subtree X_0 of X. By Definition 8.1, $\pi(X)$ is the free group on $EX - EX_0$, $\pi(X)^{ab} \approx \mathbb{Z}[EX - EX_0]$, and the natural map to $\mathbb{Z}[EX]$ takes the form $\mathbb{Z}[EX - EX_0] \to \mathbb{Z}[EX], e \mapsto e + X_0[\tau e, \iota e]$, where $X_0[-, -]$ is as in Definition 6.5. This is clearly injective, since composing with the projection onto $\mathbb{Z}[EX - EX_0]$ gives the identity.

It remains to prove exactness at $\mathbb{Z}[EX]$. Suppose $\partial\left(\sum_{e \in EX} n_e e\right) = 0$. We know that $\partial\left(\sum_{e \in EX} n_e(e + X_0[\tau e, \iota e])\right) = 0$ so $\partial\left(\sum_{e \in EX} n_e X_0[\tau e, \iota e]\right) = 0$. But ∂ is injective on $\mathbb{Z}[EX_0]$ by Lemma 6.4, so $\sum_{e \in EX} n_e X_0[\tau e, \iota e] = 0$. Hence $\sum_{e \in EX} n_e e = \sum_{e \in EX} n_e(e + X_0[\tau e, \iota e])$, which is in the image of $\pi(X)^{ab}$, as desired. ∎

We note two consequences of the case where G acts freely on X, so $P = \pi(G\backslash X)$.

9.3 Corollary. *If X is a connected G-free G-graph then there is an extension of groups $1 \to \pi(X) \to \pi(G\backslash X) \to G \to 1$.* ∎

9.4 Corollary. *If F is a free group on a set S, N a normal subgroup of F and $G = F/N$ then there is an exact sequence of G-modules*

$$0 \to N^{ab} \to \mathbb{Z}[G \times S] \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Proof. Let X be the Cayley graph for G with respect to S, so we have an extension of groups $1 \to \pi(X) \to \pi(G\backslash X) \to G \to 1$ and an exact sequence of G-modules $0 \to \pi(X)^{ab} \to \mathbb{Z}[G \times S] \to \mathbb{Z}G \to \mathbb{Z} \to 0$. Further we can identify $F = \pi(G\backslash X)$ so $N = \pi(X)$. ∎

9.5 Remark. In order to have a complete structure theorem for a group acting on a connected graph, we want an explicit description of $\pi(X)$ as a subgroup of P. An element of $\pi(X)$ corresponds to a unique closed path in X at v_0 and this can be expressed in the form

$$v_0, g_0 t_{e_1}^{\frac{1}{2}(\varepsilon_1 - 1)} e_1^{\varepsilon_1}, g_0 t_{e_1}^{\varepsilon_1} v_1, g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\frac{1}{2}(\varepsilon_2 - 1)} e_2^{\varepsilon_2}, g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} v_2, \ldots,$$
$$g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g \cdots g_{n-1} t_{e_n}^{\frac{1}{2}(\varepsilon_n - 1)} e_n^{\varepsilon_n}, g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n} v_n = v_0,$$

where $v_0, e_1^{\varepsilon_1}, v_1, e_2^{\varepsilon_2}, v_2, \ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n = v_0$ is a closed path in Y and $g_i \in G_{v_i}$ for all $i \in [0, n]$. Thdn $g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n}$ is an element of G_{v_0}, and denoting it by g_n^{-1}, we get an expression $g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n} g_n$ representing the desired element of P.

For the purpose of presenting G, one wants a set of elements which generate $\pi(X)$ as normal subgroup of P; geometrically this amounts to a set of closed paths at v_0 in X whose G-translates generate all of $\pi(X)$.

For example, if X is the 1-skeleton of a simply connected CW-complex on which G acts cellularly, respecting the orientation of X, then it suffices to take one two-cell from each G-orbit and take the corresponding elements of $\pi(X)$ to present G. The next example illustrates this. ∎

9.6 Example. Let G be the group of symmetries of a cube, so G acts on the graph X in Fig. I.4(i). For any edge e, $\{\iota e, e, \tau e\}$ is a fundamental G-transversal in X, and for definiteness we choose e as indicated.

Let r_1, r_2, r_3 be the reflections in the planes of symmetry π_1, π_2, π_3 indicated in Fig. I.4(ii), (iii), (iv), respectively. It is easy to check that $G_e = \langle r_1 | r_1^2 \rangle \approx D_1$, $G_{\tau e} = \langle r_1, r_2 | r_1^2, r_2^2, (r_1 r_2)^2 \rangle \approx D_2$, and $G_{\iota e} = \langle r_1, r_3 | r_1^2, r_3^2, (r_1 r_3)^3 \rangle \approx D_3$.

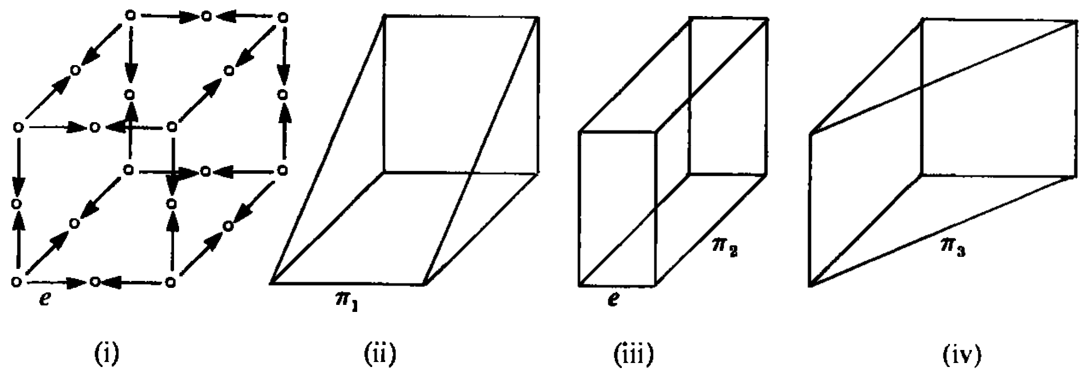


Fig. I.4

By Theorem 9.2, we have an extension of groups

$$1 \to \pi(X) \to D_3 \underset{D_1}{*} D_2 \to G \to 1;$$

here $\pi(X)$ is free of rank $1 - \chi(X) = 1 - 20 + 24 = 5$. (We remark that $D_3 \underset{D_1}{*} D_2 \approx \mathrm{PGL}_2(\mathbb{Z})$ by Example 5.2(ii), and the universal covering tree of X is as in Fig. I.3, page 22; this determines an action of $\mathrm{GL}_2(\mathbb{Z})$ on the cube.)

The path around the bottom of the cube starting with e gives a relation $(r_2r_3)^4 = 1$, as in Example 2.2(v). The G-translates of this path generate all of $\pi(X)$, and we arrive at the presentation

$$G = \langle r_1, r_2, r_3 \mid r_1^2, r_2^2, r_3^2, (r_1r_2)^2, (r_1r_3)^3, (r_2r_3)^4 \rangle.$$

It is evident that $|G| = |G_e|\,|Ge| = 2 \times 24 = 48$.

In fact, it can be seen by the action on the four diagonals of the cube that $G \approx C_2 \times \mathrm{Sym}\,4$.

The cube has an obvious simply connected CW-structure with X as one-skeleton. There is exactly one G-orbit of faces, and in the above argument the bottom face is chosen as a representative of the G-orbit to find a suitable element generating $\pi(X)$ as normal subgroup of $D_3 \underset{D_1}{*} D_2$.

∎

10 Free products

10.1 Notation. For any set E and equivalence relation $S \subseteq E \times E$ we denote the set of equivalence classes by E/S. The equivalence relation for equality is denoted $\Delta E = \{(e, e) \mid e \in E\}$, called the *diagonal*.

Let X be a graph and $S \subseteq EX \times EX$ an equivalence relation on EX. We define VS to be the equivalence relation on VX generated by $\{(\iota e, \iota f), (\tau e, \tau f) \mid (e, f) \in S\}$. Then the incidence functions $\iota, \tau: EX \to VX$

induce functions $\bar{\iota}, \bar{\tau}: EX/S \to VX/VS$. We denote by X/S the resulting graph. Notice there is a surjective graph map $X \to X/S$.

10.2 Lemma. *If T is a tree and $S \subseteq ET \times ET$ an equivalence relation on ET such that for all $(e,f)\in S$ either $\iota e = \iota f$ or $\tau e = \tau f$ then T/S is a tree.*

Proof. Since the statements concern only finitely many elements at a time, we may assume that T is finite.

Consider now the case where $S = \Delta EX \cup \{(e,f),(f,e)\}$ for distinct edges e, f of T such that $\iota e = \iota f$. Consider the graph $T - \{f\}$ obtained by deleting f from T. Here τe and τf lie in the two different components, so identifying τe and τf attaches together the two components, and we get a tree. It is not difficult to see that this graph is isomorphic to T/S, so T/S is a tree.

The general case is obtained by a finite repetition of such constructions, and the result follows by induction. ∎

10.3 Theorem. *Let N be a normal subgroup of G, and $\bar{T}$ a G/N-tree such that $E\bar{T}$ is G/N-free. If there exists a G-tree T with G-free edge set, and a surjective map of G-trees $T \to \bar{T}$, then there exists such a T with $N\backslash T \approx \bar{T}$ as G-trees.*

Proof. Let us denote the map $T \to \bar{T}$ by $t \mapsto \bar{t}$. Here we have a surjective map of G/N-graphs, $N\backslash T \to \bar{T}$, $Nt \mapsto \bar{t}$. Suppose it is not injective. Since $\bar{T}$ is a tree, it follows that $N\backslash ET \to E\bar{T}$, $Ne \mapsto \bar{e}$, is not injective. This is a map of free G/N-sets, so two G/N-orbits get identified. Thus there exist $e, f \in ET$ with $Ge \neq Gf$, $\bar{e} = \bar{f}$. Hence we can construct a path $p = e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ in T with $Ge_1 \neq Ge_n$, $\bar{e}_1 = \bar{e}_n$.

We claim we can choose such a path p with $n = 2$. Clearly $n \geqslant 2$. Since $\bar{p} = \bar{e}_1^{\varepsilon_1}, \ldots, \bar{e}_n^{\varepsilon_n}$ is a path in $\bar{T}$ with $\bar{e}_1 = \bar{e}_n$, there is some $i \in [1, n-1]$ with $\bar{e}_{i+1}^{\varepsilon_{i+1}} = \bar{e}_i^{-\varepsilon_i}$. If $Ge_{i+1} \neq Ge_i$ then we can take $p = e_i^{\varepsilon_i}, e_{i+1}^{\varepsilon_{i+1}}$ and achieve $n = 2$. It remains to consider the case $Ge_{i+1} = Ge_i$. Here $n \geqslant 3$ and $e_{i+1} = ge_i$ for some $g \in G$. If $i = 1$ or $n-1$ we can delete the first or last edge of p, respectively, and reduce n by 1. Thus we may assume $n \geqslant 4$ and $i \in [2, n-2]$. Since $\bar{e}_{i+1} = g\bar{e}_i = g\bar{e}_{i+1}$ and $E\bar{T}$ is G/N-free we see $g \in N$. Thus $g\bar{e}_1 = \bar{e}_1 = \bar{e}_n$ and $Gge_1 = Ge_1 \neq Ge_n$, while the path $p' = ge_1^{\varepsilon_1}, \ldots, ge_i^{\varepsilon_i}, e_{i+1}^{\varepsilon_{i+1}}, \ldots, e_n^{\varepsilon_n}$ in T is not a reduced path. Hence we can delete $ge_i^{\varepsilon_i}, e_{i+1}^{\varepsilon_{i+1}}$ from p' and reduce n. This proves the claim that we may assume $n = 2$.

Here $e_1^{\varepsilon_1}, e_2^{\varepsilon_2}$ is a path in T with $\bar{e}_1 = \bar{e}_2, Ge_1 \neq Ge_2$. Since $\bar{e}_1^{\varepsilon_1}, \bar{e}_2^{\varepsilon_2}$ is a path in $\bar{T}$ with $\bar{e}_1 = \bar{e}_2$ we see that $\varepsilon_1 = -\varepsilon_2$. Since $e_1^{\varepsilon_1}, e_2^{\varepsilon_2}$ is a path in T either $\iota e_1 = \iota e_2$ or $\tau e_1 = \tau e_2$. Let $S = \Delta E \cup \{(ge_1, ge_2), (ge_2, ge_1) | g \in G\}$. It is easy to see that this is an equivalence relation on EX since $Ge_1 \neq Ge_2$ and e_1, e_2 have trivial stabilizers. By Lemma 10.2, it follows that the G-graph $T' = T/S$ is a tree. Further T' has G-free edge set and there is a surjective map of G-trees $T' \to \bar{T}$.

By Zorn's Lemma there is a largest equivalence relation S on ET such that the graph $T' = T/S$ is still a G-tree with G-free edge set having a surjective map of G-trees $T' \to \bar{T}$. The preceding argument shows that, by maximality, $N\backslash T' \to \bar{T}$ must be an isomorphism, as desired. ∎

10.4 Theorem. *Let I be a set and $(\alpha_i: G_i \to \bar{G}_i \mid i \in I)$ a family of surjective group homomorphisms. Denote the free products by $G = \ast_{i\in I} G_i$, $\bar{G} = \ast_{i\in I} \bar{G}_i$, and the resulting map by $\alpha: G \to \bar{G}$. For any subgroup H of G such that $\alpha(H) = \bar{G}$, there exist subgroups H_i, $i \in I$, of H such that $\alpha(H_i) = \bar{G}_i$ and $H = \ast_{i\in I} H_i$.*

Proof. Clearly we may assume that I is nonempty. Let Y be a tree with vertex set I, and incidence functions ι, τ; for example, Y can be a star with one specified element of I being the initial vertex of each edge.

Let $(G(-), Y)$ be the graph of groups such that, for all $i \in VY = I$, $G(i) = G_i$, and for all $e \in EY$, $G(e) = 1$ so $G(e) \subseteq G(\iota e)$ and there is a unique group homomorphism $t_e: G(e) \to G(\tau e)$. Then $G = \pi(G(-), Y, Y)$ acts on the tree $T = T(G(-), Y, Y)$. Here ET is G-free.

Define $(\bar{G}(-), Y)$ and $\bar{T} = T(\bar{G}(-), Y)$ similarly, so $E\bar{T}$ is $\bar{G}$-free.

There is a surjective graph map $T \to \bar{T}$; in fact, $\bar{T} = (\operatorname{Ker}\alpha)\backslash T$.

We view $T, \bar{T}$ as H-trees in the obvious way. Then $T \to \bar{T}$ is a surjective map of H-trees, ET is H-free and $E\bar{T}$ is H/N-free, where $N = H \cap \operatorname{Ker}\alpha$.

By Theorem 10.3 there exists an H-tree T' with H-free edge set such that $N\backslash T' = \bar{T}$. The natural copy of Y in $\bar{T}$ is a $\bar{G}$-transversal, and $\bar{G} = \alpha(H) = H/N$. Thus we can lift the tree Y in $N\backslash T'$ back to a subtree Z in T' and Z is a fundamental H-transversal in T'. Since ET' is H-free, the Structure Theorem 4.1 shows that $H = \ast_{v\in VZ} H_v$. Further, for each $v \in VZ$, $\bar{G}_{\bar{v}} = \alpha(H)_v = \alpha(H_{\bar{v}}) = \alpha(NH_v) = \alpha(H_v)$, as desired. ∎

10.5 Theorem. *Let I be a set, and $(G_i \mid i \in I)$ a family of groups. For any free group F and surjective homomorphism $\alpha: F \to \ast_{i\in I} G_i$, there exist subgroups F_i, $i \in I$, of F such that $F = \ast_{i\in I} F_i$ and $\alpha(F_i) = G_i$.*

Proof. Consider the natural map $\beta: \ast_{i\in I} \alpha^{-1}(G_i) \to F$. The image of β is the subgroup of F generated by the $\alpha^{-1}(G_i)$, $i\in I$, which is $\alpha^{-1}(G)=F$. As β is a surjective map to a free group, there exists a homomorphism $\gamma:F\to \ast_{i\in I}\alpha^{-1}(G_i)$ such that $\beta\gamma$ is the identity on F; in particular, γ is injective. Hence we have a copy γF of F in $\ast_{i\in I}\alpha^{-1}(G_i)$ and it is easy to check that the restriction of the natural surjection $\ast_{i\in I}\alpha^{-1}(G_i)\to \ast_{i\in I} G_i$ to the copy of F is the given surjection α. The result now follows from Theorem 10.4. ∎

10.6 The Grushko–Neumann Theorem. *For any set I and family $(G_i|i\in I)$ of groups*, $\operatorname{rank}(\ast_{i\in I} G_i)=\sum_{i\in I}\operatorname{rank} G_i$.

Proof. It is clear that $\operatorname{rank}(\ast_{i\in I} G_i)\leqslant \sum_{i\in I}\operatorname{rank} G_i$.

Let F be a free group with $\operatorname{rank}(F)=\operatorname{rank}(\ast_{i\in I} G_i)$, so there is a surjection $\alpha:F\to \ast_{i\in I} G_i$. By Theorem 10.5, there are subgroups F_i, $i\in I$, of F such that $F=\ast_{i\in I} F_i$ and $\alpha(F_i)=G_i$, so $\sum_{i\in I}\operatorname{rank} G_i\leqslant\sum_{i\in I}\operatorname{rank} F_i$. But the F_i are free groups by the Nielsen–Schreier Theorem 8.4, so $\sum_{i\in I}\operatorname{rank} F_i=\operatorname{rank} F=\operatorname{rank}(\ast_{i\in I} G_i)$, and we have the reverse inequality. ∎

Notes and comments

The book of Serre (1977), written with the collaboration of Bass, forms the foundation of this chapter.

The first three sections contain little more than notation, not all of which is standard, as noted by Rota (1986). For example, what we call a *graph* is usually called a directed multigraph, and what we call a *tree* is usually called an oriented tree. We felt at liberty to give the short names to the concepts which occurred most frequently in this work.

Free products with amalgamation were introduced by Schreier (1927); the HNN extension take its name from the initials of the authors of Higman, Neumann and Neumann (1949), where the concept was first studied.

Section 4 is taken from Serre (1977). The Structure Theorem 4.1 is due to Bass and Serre; the proof can be used to obtain a normal form, which in turn can be used as a rather cumbersome substitute for a tree. Theorem 4.12 is due to Tits, and Proposition 4.7 is classical.

In Example 5.2, the tree is from Serre (1977), where the action of $SL_2(\mathbb{Z})$ is described; the extension to $GL_2(\mathbb{Z})$ was pointed out to us by Paul Gerardin. The group-theoretic conclusions are essentially well-known.

Example 5.3 is due to Serre (1977), and the interested reader can find many more details there. The group-theoretic conclusions in (iii), (ii) are a theorem of Nagao (1959) and a generalization of a theorem of Ihara (1966), respectively.

Example 5.4 is a condensed version of the survey we wrote for Cohn (1985); the interested reader will find there details of the arguments, and the numerous attributions.

Theorem 7.6 is due to Bass and Serre, see Serre (1977), and the proof here follows Dicks (1980). Other, more topological, proofs can be found in Chiswell (1979), and Scott and Wall (1979). Theorem 7.7 is a generalization of a result of H. Neumann (1948), which in turn generalizes Theorem 7.8, of Kurosh (1937).

Theorem 8.2 has long been known. Theorem 8.3 seems to have been first stated explicitly in Serre (1977), but is essentially contained in Reidemeister (1932), Section 4, 20. Nielsen (1921) proved Theorem 8.4 for finitely generated subgroups, and Schreier (1927) proved the general case and Theorem 8.5.

Theorem 8.6 is due to Gersten (1984); the elegant proof given here is extracted from Goldstein and Turner (1986) where more is proved; Conjecture 8.7 is due to G.P. Scott.

Theorem 8.8 is due to Howson (1954), whose proof mentions trees. Conjecture 8.9 is from H. Neumann (1953).

Theorem 9.2 is from Serre (1977), and the explicit version in Remark 9.5 was written out by Brown (1984). Corollary 9.4 is due to Lyndon (1950).

Theorem 10.3 is based on results of Chiswell (1976) and Stallings (1965). Theorem 10.4 is due to Higgins (1966). Theorem 10.5 goes back to Wagner (1957), and the argument given here was shown to us by E. Formanek. Theorem 10.6 was proved independently by Grushko (1940) and B.H. Neumann (1943).

II

Cutting graphs and building trees

Section 1 gives a useful characterization of trees in terms of vertices acting as functions on the edge set. Section 2 introduces the concept of the Boolean ring of a connected G-graph, and associates with it an inverse limit of G-trees. This is used in Section 3 to determine the infinite finite-valency distance-transitive graphs, and will also be important in the next chapter.

1 Tree sets

We begin with terminology and notation which will be used frequently throughout the chapter.

1.1 Definitions. Let E, V be G-sets and A a nonempty set.

(i) The set of all functions from E to A will be denoted (E, A); this is a G-set with $(gv)(e) = v(g^{-1}e)$ for all $v \in (E, A)$, $g \in G$, $e \in E$.

If there is specified a G-map $V \to (E, A)$ we denote it by $v \mapsto v|E$ and write $V|E$ for the image. The value of $v|E$ at e will be denoted simply $v(e)$. There is then a dual G-map $E \to (V, A)$, denoted $e \mapsto e|V$, and the same notation applies; thus for $e \in E$, $v \in V$, $e(v) = v(e)$.

(ii) Since $\mathbb{Z}_2$ has a ring structure, $(V, \mathbb{Z}_2)$ is a ring under pointwise addition and multiplication. The 0 and 1 are the obvious constant functions.

If $b \in (V, \mathbb{Z}_2)$ then b^* denotes $1 - b$, or equivalently $1 + b$. For any subset F of $(V, \mathbb{Z}_2)$, F^* denotes $\{f^* | f \in F\}$.

Let $a, b \in (V, \mathbb{Z}_2)$. If $ab^* = 0$ we write $a \leqslant b$; this defines a partial order on $(V, \mathbb{Z}_2)$. We denote by $a \square b$ the family consisting of ab, a^*b, ab^*, a^*b^*; these are then four distinct elements of $a \square b$ even though some of them may be equal as elements of $(V, \mathbb{Z}_2)$. We say that a and b are *nested* if