Math 8211 **Commutative and Homological Algebra I** **Fall 2021**

Homework Assignment 1 Due Thursay 9/23/2021.

Eisenbud defines an irreducible element r to be a non-unit for which $r = st$ with $s, t \in R$ implies one of s and t is a unit. We will work with Eisenbud's definition that R is a UFD if it is an integral domain and elements of R can all be factored uniquely as a finite product of irreducible elements (look at the book to see what 'uniquely' means). We define an element r to be prime if and only if it is not a unit and whenever r divides st then r divides s or r divides t, where 'r divides u' means $u = rr'$ for some element r'.

1. Show that an element r is prime if and only if the ideal (r) is a prime ideal.

2. Show that prime elements are always irreducible.

3. Show that, in a UFD, irreducible elements are prime.

4. Show that a domain R that satisfies both the following conditions must be a UFD:
(a) irreducible elements are prime,
(b) every element is a finite product of irreducible elements.

5. (i) Let r be an element of an integral domain R. Show that r is irreducible if and only if (r) is maximal among proper principal ideals.
(ii) Show that condition (b) of question 4. is equivalent to the condition that R has the ascending chain condition on principal ideals.

6. Eisenbud Exercise 1.1 on page 46

7. Eisenbud Exercise 1.9 on page 49

8. Eisenbud 1.24 ((a) and (b) only) on page 55

EXTRA QUESTIONS: DO NOT HAND IN

9. Let $A_1, \ldots, A_n$ be rings. Show that the prime ideals of $A_1 \times \ldots \times A_n$ are the ideals of the form $A_1 \times \cdots A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n$ where P_i is a prime ideal of A_i.

10. Let $\{P_\lambda \mid \lambda \in \Lambda\}$ be a non-empty family of prime ideals, and suppose they are totally ordered by inclusion. Show that $\bigcap_{\lambda \in \Lambda} P_\lambda$ is a prime ideal.

11. Eisenbud 1.2

12. Eisenbud 1.13