Math 8211 **Commutative and Homological Algebra I** **Fall 2021**

Homework Assignment 3 Due Saturday 11/27/2021, uploaded to Gradescope.

1. Let k be a field. A monomial ideal I of $k[x_0, \ldots, x_r]$ is defined to be an ideal generated by monomials. Show that such an ideal. I has a basis over k consisting of the monomials it contains, and that the monomials not in I span a subspace W of $k[x_0, \ldots, x_r]$ so that $k[x_0, \ldots, x_r] = I \oplus W$.

2. (Eisenbud 3.6) Eisenbud characterizes prime monomial ideals as ideals 'generated by subsets of the variables,' irreducible monomial ideals as ideals 'generated by powers of some of the variables,' radical monomial ideals as 'ideals generated by square-free monomials,' and primary monomial ideals as ideals 'containing a power of each of a certain subset of the variables, and generated by elements involving no further variables.'
(a) Prove Eisenbud's characterization of prime monomial ideals.
(b) Prove Eisenbud's characterization of irreducible monomial ideals.
(c) Prove Eisenbud's characterization of radical monomial ideals.
(d) Prove Eisenbud's characterization of primary monomial ideals.

3. (Eisenbud 3.8) Find an algorithm for computing an irreducible decomposition, and thus a primary decomposition, of a monomial ideal.

4. Find an irreducible decomposition, and also two different minimal primary decompositions of the ideal (x^4y, x^2y^2, y^3) in $k[x, y]$.

5. (a) Show that the radical of a primary ideal is prime.
(b) If I is a proper ideal containing a power $\mathfrak{m}^n$ of a maximal ideal $\mathfrak{m}$ show that I is primary and $\mathrm{rad}(I) = \mathfrak{m}$.
(c) Find an example of a power of a prime ideal that is not primary.

6. If P is a prime ideal of R the symbolic nth power of P is the ideal $P^{(n)} = P^n R_P \cap R$. Show that this is a primary ideal with radical P. (To assign meaning to the intersection, assume that R is a domain.)

Extra questions: do not upload to Gradescope.

7. (Eisenbud 3.7) Find an algorithm for computing the radical of a monomial ideal.

8. Let. I and J be ideals of a Noetherian ring R. Prove that if $JR_P \subset IR_P$ for every. $P \in \mathrm{Ass}_R(R/I)$ then $J \subset I$.

9. Let R be a Noetherian ring and let $x \in R$ be an element which is neither a unit nor a zero-divisor. Prove that $\mathrm{Ass}_R(R/xR) = \mathrm{Ass}_R(R/x^nR)$.