**Math 8211** **Commutative and Homological Algebra I** **Fall 2021**

**Homework Assignment 4** Due Saturday 12/18/2021, uploaded to Gradescope.

1. Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of $R$-modules, then for every $n \geq 0$ the sequence $0 \to \mathrm{Ext}^n_R(D, L) \to \mathrm{Ext}^n_R(D, M) \to \mathrm{Ext}^n_R(D, N) \to 0$ is also short exact and split. [Use a splitting homomorphism and the fact that Ext is functorial in each variable.]

2. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of right $R$-modules where both $A$ and $C$ are flat. Prove that $B$ is flat.

3. (a) Suppose that $U$, $V$, and $W$ are $R$-modules and that there are homomorphisms

$$U \underset{\delta}{\overset{\alpha}{\rightleftarrows}} V \underset{\gamma}{\overset{\beta}{\rightleftarrows}} W$$

such that $\beta\alpha = 0$ and such that the identity map on $V$ can be written $1_V = \alpha\delta + \gamma\beta$. Show that $\beta = \beta\gamma\beta$. Suppose in addition to all this that $\alpha = \alpha\delta\alpha$. Show that $V \cong \alpha\delta(V) \oplus \gamma\beta(V)$.
(b) Recall that a chain complex $C$ of $R$-modules is called *contractible* if it is chain homotopy equivalent to the zero chain complex. Prove that $C$ is contractible if and only if $C$ can be written as a direct sum of chain complexes of the form $\cdots \to 0 \to A \xrightarrow{\alpha} B \to 0 \cdots$ where $\alpha$ is an isomorphism.

4. Let $R = k[X]/(X^3)$ where $k$ is a field. Let $C$ be the complex $R \xrightarrow{X^2} R$.
(a) Find $\dim_k \mathrm{Hom}(C, C)$, the dimension of the space of chain maps from $C$ to $C$.
(b) Find the dimension of the subspace of chain maps $C \to C$ which are homotopic to zero. Hence find the dimension of the space $\underline{\mathrm{Hom}}(C, C)$ of homotopy classes of chain maps $C \to C$.

Extra question parts for question 4: do **not** hand in parts (c), (d), (e) or (f).
(c) Show that, for this complex $C$, the set of chain maps $C \to C$ that are non-isomorphisms forms a vector subspace of the space of all endomorphisms of $C$. Find the dimension of this subspace.
(d) Show that it is possible to find another complex $D$ for which the set of non-isomorphisms $D \to D$ does not form a vector subspace of all endomorphisms.
(e) Show that, for this complex $C$, all chain maps $C \to C$ which are equivalences are, in fact, automorphisms
(f) Determine, for this complex $C$, whether or not all invertible chain maps $C \to C$ are homotopic to each other.

5. Given a homomorphism of chain complexes of $R$-modules $\phi : \mathcal{C} \to \mathcal{D}$ we may define $E_n = C_{n-1} \oplus D_n$, and a mapping $e_n : E_n \to E_{n-1}$ by $e_n(a, b) = (-\partial a, \phi a + \partial b)$, where we denote the boundary maps on $\mathcal{C}$ and $\mathcal{D}$ by $\partial$. The specification $\mathcal{E}(\phi) = \{E_n, e_n\}$ is called the *mapping cone* of $\phi$.

(a) Show that $\mathcal{E} = \{E_n, e_n\}$ is indeed a chain complex.
(b) Show that there is a short exact sequence of chain complexes $0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{C}[1] \to 0$ where $\mathcal{C}[1]$ denotes the chain complex with the same $R$-modules and boundary maps as $\mathcal{C}$ but with the labeling of degrees shifted by 1 in an appropriate direction. Deduce that there is a long exact sequence

$$\cdots \to H_n(\mathcal{C}) \to H_n(\mathcal{D}) \to H_n(\mathcal{E}(\phi)) \to H_{n-1}(\mathcal{C}) \to \cdots$$

(c) Show that $\mathcal{E}(\phi)$ is acyclic if and only if $\phi$ induces an isomorphism $H_n(\mathcal{C}) \to H_n(\mathcal{D})$ for every $n$.
Extra question part: do **not** hand in part (d).
(d) Show that if $\phi \simeq \psi : \mathcal{C} \to \mathcal{D}$ then $\mathcal{E}(\phi) \cong \mathcal{E}(\psi)$.

6. (a) Suppose that we have chain maps $C \xrightarrow{f} D \xrightarrow{g} E$ and suppose that $D$ is a contractible complex. Show that the composite $gf$ is homotopic to zero (i.e. null homotopic).
(b) Consider the diagram

$$\begin{array}{ccccccccccc}
C: & \cdots & \xrightarrow{d} & C_2 & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_0 & \xrightarrow{d} & \cdots \\
\downarrow{\scriptstyle i_C} & & & \downarrow{\scriptstyle \binom{d}{1}} & & \downarrow{\scriptstyle \binom{d}{1}} & & \downarrow{\scriptstyle \binom{d}{1}} & & \\
I_C: & \cdots & \xrightarrow{\delta} & C_1 \oplus C_2 & \xrightarrow{\delta} & C_0 \oplus C_1 & \xrightarrow{\delta} & C_1 \oplus C_0 & \xrightarrow{\delta} & \cdots
\end{array}$$

where $\delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Show that $I_C$ is contractible and that $i_C$ is a one-to-one chain map.
(c) Show that if $f = Td + eT : C \to D$ is any null-homotopic map of complexes then $f$ defines a chain map $I_C \to D$ as follows:

$$\begin{array}{ccccccccccc}
I_C: & \cdots & \xrightarrow{\delta} & C_1 \oplus C_2 & \xrightarrow{\delta} & C_0 \oplus C_1 & \xrightarrow{\delta} & C_1 \oplus C_0 & \xrightarrow{\delta} & \cdots \\
\downarrow & & & \downarrow{\scriptstyle (T,eT)} & & \downarrow{\scriptstyle (T,eT)} & & \downarrow{\scriptstyle (T,eT)} & & \\
D: & \cdots & \xrightarrow{e} & D_2 & \xrightarrow{e} & D_1 & \xrightarrow{e} & D_0 & \xrightarrow{e} & \cdots
\end{array}$$

such that the composite of this morphism with $i_C$ is $f$. Deduce that any null-homotopic map factors through a contractible complex.

7. Show that the two extensions $0 \to \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/3\mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \xrightarrow{\mu'} \mathbb{Z} \xrightarrow{\epsilon'} \mathbb{Z}/3\mathbb{Z} \to 0$ are not equivalent, where $\mu = \mu'$ is multiplication by 3, $\epsilon(1) \equiv 1 \pmod 3$ and $\epsilon'(1) \equiv 2 \pmod 3$.

**Extra questions: do not upload to Gradescope.**

8. Let $A$ be an abelian group. Prove that $A$ is free abelian if and only if $\mathrm{Ext}^1_{\mathbb{Z}}(A, F) = 0$ for every free abelian group $F$.

9. Show that in any commutative diagram of $R$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

in which the right hand vertical morphism is the identity and the rows are exact, the left hand square is necessarily a pushout. Also the dual statement.

10. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $R$-modules. Show that in the long exact sequence

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \to \mathrm{Hom}(C, C) \xrightarrow{\delta} \mathrm{Ext}^1(C, A) \to \cdots$$

the image of $1_C$ under the connecting homomorphism $\delta$ is the Ext class of the extension.

11. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables over a field $k$. Let us regard $k$ as the unital $R$-module on which all of $x_1, \ldots, x_n$ act as 0.
(a) Show that $\dim_k \mathrm{Ext}^1_R(k, k) = n$
(b) Let $0 \to k^n \to E \to k \to 0$ be an extension of $R$-modules whose Ext class, when written in terms of components with respect to the direct sum decomposition $\mathrm{Ext}^1_R(k, k^n) \cong \bigoplus_{i=1}^n \mathrm{Ext}^1_R(k, k)$, has components which are a basis of $\mathrm{Ext}^1_R(k, k)$. Show that $k^n$ is the unique maximal submodule of $E$ and that $E$ is indecomposable as an $R$-module (i.e. $E$ cannot be expressed as a direct sum of two non-zero submodules). Show that $E$ is isomorphic to $R/(x_1, \ldots, x_n)^2$.
(c) Show that any extension of the form $0 \to k^{n+1} \to E' \to k \to 0$ must have a module $E'$ in the middle which decomposes as an $R$-module.