Homework Assignment 2 Due Saturday 3/5/2022, uploaded to Gradescope.
Each question part is worth 1 point. There are 12 question parts. Assume that all categories are small. We define $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to be the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.

1. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories.

(a) Show that, for all objects $x, y \in \mathrm{Ob}\mathcal{C}$, the functor F provides a bijection

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \leftrightarrow \mathrm{Hom}_{\mathcal{D}}(F(x), F(y)),$$

that preserves composition, so that $\mathrm{End}_{\mathcal{C}}(x) \cong \mathrm{End}_{\mathcal{D}}(F(x))$ as monoids.

(b) Show that $x \cong y$ in $\mathcal{C}$ if and only if $F(x) \cong F(y)$ in $\mathcal{D}$, so that F provides a bijection between the isomorphism classes of $\mathcal{C}$, and of $\mathcal{D}$.

(c) Let $\mathcal{E}$ be a further category. Show that the functor categories $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{E})$ are naturally equivalent.

2. Let $\mathcal{C}$ be a category and let $x, y \in \mathrm{Ob}\mathcal{C}$. Prove that if $x \cong y$ then $\mathrm{Hom}_{\mathcal{C}}(x, -)$ and $\mathrm{Hom}_{\mathcal{C}}(y, -)$ are naturally isomorphic functors $\mathcal{C} \to \mathrm{Set}$.

3. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors and $\eta : F \to G$ a natural transformation.

(a) Show that if, for all $x \in \mathrm{Ob}\mathcal{C}$, the mapping $\eta_x : F(x) \to G(x)$ is an isomorphism in $\mathcal{D}$, then η is a natural isomorphism (meaning that it has a 2-sided inverse natural transformation $\theta : G \to F$).

(b) Suppose that F is an equivalence of categories and that F is naturally isomorphic to G, so $F \simeq G$. Show that G is an equivalence of categories.

4. Let G be a group, which we regard as a category $\mathcal{G}$ with a single object, and with the elements of G as morphisms. Let $F : \mathcal{G} \to \mathcal{G}$ be a functor.

(a) Show that F is naturally isomorphic to the identity functor $1_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ if and only if the mapping $F : G \to G$, induced by F on the set of morphisms, is an inner automorphism; that is, an automorphism of the form $c_g : G \to G$ for some $g \in G$, where $c_g(h) = ghg^{-1}$ for all $h \in G$.

(b) Show that self equivalences of $\mathcal{G}$ are automorphisms of $\mathcal{G}$.

(c) Show that the group of natural isomorphism classes of self equivalences of $\mathcal{G}$ is isomorphic to $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. (In the context of group theory, $\mathrm{Inn}(G)$ denotes the set of inner automorphisms of G, and $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the group of *outer* (or *non-inner*) automorphisms.)

5. Let I be the poset with two elements 0 and 1, and with $0 < 1$. If P and Q are posets we can regard them as categories $\mathcal{P}$ and $\mathcal{Q}$ whose objects are the elements of the posets, and where there is a unique morphism $x \to y$ if and only if $x \leq y$.

(a) Show that if P and Q are posets then a functor $\mathcal{P} \to \mathcal{Q}$ is 'the same thing as' an order-preserving map. (Don't worry about any fancy interpretation of 'the same thing as'!)

(b) Now consider two functors $F, G : \mathcal{P} \to \mathcal{Q}$, which we may regard as order-preserving maps $f, g : P \to Q$ by part (a). Show that the following three conditions are equivalent:
(i) there exists a natural transformation $F \to G$,
(ii) $f(x) \leq g(x)$ for all $x \in P$,
(iii) there is an order-preserving map $h : P \times I \to Q$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in \mathcal{P}$. Here $P \times I$ denotes the product poset with order relation $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$, where $a_i \in P$ and $b_i \in I$.

6. Let $1_{R-\text{mod}} : R\text{-mod} \to R\text{-mod}$ denote the identity functor. Let $\text{Nat}(1_{R-\text{mod}}, 1_{R-\text{mod}})$ denote the set of natural transformations from this functor to itself, noting that this set has the structure of a ring (multiplication is composition and addition comes because we can add homomorphisms of R-modules, so that for two natural transformations θ, ψ at an object x we have $(\theta + \psi)_x = \theta_x + \psi_x$). Show that $\text{Nat}(1_{R-\text{mod}}, 1_{R-\text{mod}}) \cong Z(R)$.

Extra question: do not upload to Gradescope.

7. Let $\mathcal{C}$ be a small category and let $F, G : \mathcal{C} \to \text{Set}$ be functors. Show that a natural transformation of functors $\tau : F \to G$ is an epimorphism in $\text{Fun}(\mathcal{C}, \text{Set})$ if and only if for every object x of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a surjection; and it is a monomorphism if and only if for every object x of $\mathcal{C}$, $\tau_x : F(x) \to G(x)$ is a 1-1 map.

8. Write out a proof that if G is the right adjoint of a functor F with the property that F preserves monomorphisms, then G sends injective objects to injective objects.

9. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors with F left adjoint to G, and with adjunction unit η and counit ϵ. Write out a proof that the second triangular identity holds, namely the following triangle commutes:

$$\begin{array}{ccccc} G & & \xrightarrow{1_G} & & G \\ & \underset{\eta_G}{\searrow} & & \underset{G\epsilon}{\nearrow} & \\ & & GFG & & \end{array}$$

10. Assume the axiom of choice in this question, or else make some assumption such as: everything is finite. Let $\mathcal{C}$ be a category, and for each isomorphism class $\hat{x}$ of objects x, choose a fixed representative $u_{\hat{x}}$. For each object x choose a fixed isomorphism $i_x : x \to u_{\hat{x}}$. Let $\mathcal{D}$ be the full subcategory whose objects are the $u_{\hat{x}}$ where $x \in \text{Ob}\mathcal{C}$. 'Full' means that

for each pair of objects y, z of $\mathcal{D}$ we have $\mathrm{Hom}_{\mathcal{D}}(y,z) = \mathrm{Hom}_{\mathcal{C}}(y,z)$. Define $F(x) = \hat{x}$, and for each morphism $\alpha : x \to y$ define $F(\alpha) : F(x) \to F(y)$ to be $i_y \alpha i_x^{-1}$.

(a) Show that F is a functor.

(b) Show that F and the inclusion functor $\mathrm{inc} : \mathcal{D} \to \mathcal{C}$ are inverse equivalences of categories $\mathcal{D} \simeq \mathcal{C}$. (It will help to assume that when $x = u_{\hat{x}}$, the chosen isomorphism is the identity 1_x.)

(c) Deduce that the category Set of finite sets is equivalent to the category with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$ and where $\mathrm{Hom}(n, m)$ is the set of all mappings of sets from $\mathbf{n} := \{1, \ldots, n\}$ to $\mathbf{m} := \{1, \ldots, m\}$. We take $\mathbf{0} = \emptyset$.

(d) Deduce also the following: let K be a field. Show that the category Vec of finite dimensional vectors spaces over K is equivalent to the category $\mathcal{C}$ with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$, where $\mathrm{Hom}_{\mathcal{C}}(n, m)$ is the set $M_{m,n}(K)$ of $m \times n$ matrices with entries in K, and where composition of morphisms is matrix multiplication. In case m or n is zero, give a definition of $\mathrm{Hom}_{\mathcal{C}}(n, m)$ that will make this question make sense.

11. Let $\mathcal{C}$ be a small category. A *self-equivalence* of $\mathcal{C}$ is an equivalence of categories $F : \mathcal{C} \to \mathcal{C}$. Show that the set of natural isomorphism classes of self equivalences of $\mathcal{C}$ is a group, with multiplication induced by composition of functors.