Math 8300 **Homework 2** **due Thursday 10/28/2021**
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1. Let A be a ring with a 1, and let V be an A-module. An element e in any ring is called *idempotent* if and only if $e^2 = e$.
(a) Show that an endomorphism $e : V \to V$ is a projection onto a subspace W if and only if e is idempotent as an element of $\mathrm{End}_A(V)$. (The term *projection* means a linear mapping onto a subspace that is the identity on restriction to that subspace.)
(b) Show that direct sum decompositions $V = W_1 \oplus W_2$ as A-modules are in bijection with expressions $1 = e + f$ in $\mathrm{End}_A(V)$, where e and f are idempotent elements with $ef = fe = 0$. (In case $ef = fe = 0$, e and f are called *orthogonal.*)

2. Consider a ring with identity that is the direct sum (as a ring) of non-zero subrings $A = A_1 \oplus \cdots \oplus A_r$.
(a) Writing $1_A = u_1 + \cdots + u_r$ with $u_i \in A_i$, show that the elements u_i are idempotent.
(b) Suppose that A has exactly n isomorphism types of simple modules. Show that $r \leq n$.

3. Let g be any non-identity element of a group G. Show that G has a simple complex character χ for which $\chi(g)$ has negative real part.

4. Suppose that V is a representation of G over $\mathbb{C}$ for which $\chi_V(g) = 0$ if $g \neq 1$. Show that $\dim V$ is a multiple of $|G|$. Deduce that $V \cong \mathbb{C}G^n$ for some n. Show that if W is any representation of G over $\mathbb{C}$ then $\mathbb{C}G \otimes_{\mathbb{C}} W \cong \mathbb{C}G^{\dim W}$ as $\mathbb{C}G$-modules.

5. Show that if every element of a finite group G is conjugate to its inverse, then every character on G is real-valued.
Conversely, show that if every character on G is real-valued, then every element of G is conjugate to its inverse.
[Extra irrelevant information: it is possible to have a group G in which every element is conjugate to its inverse, but not every complex representation of G is equivalent to a real representation.]

6. Let G permute a set Ω and let $R\Omega$ denote the permutation representation of G over R determined by Ω. This means $R\Omega$ has a basis in bijection with Ω and each element $g \in G$ acts on $R\Omega$ by permuting the basis elements in the same way that g permutes Ω.
(a) Show that when H is a subgroup of G and $\Omega = G/H$ is the set of left cosets of H in G, the kernel of G in its action on $R\Omega$ is H if and only if H is normal in G.
(b) Show that the normal subgroups of G are precisely the subgroups of the form $\mathrm{Ker}\,\chi_{i_1} \cap \cdots \cap \mathrm{Ker}\,\chi_{i_t}$ where $\chi_1, \ldots, \chi_n$ are the simple characters of G. Deduce that the normal subgroups of G are determined by the character table of G.
(c) Show that G is a simple group if and only if for every non-trivial simple character χ and for every non-identity element $g \in G$ we have $\chi(g) \neq \chi(1)$.

7. While walking down the street you find a scrap of paper with the following character table on it:

	1		1	
	1		-1	
$\cdots$	2	$\cdots$	-1	$\cdots$
	3		1	
	3		-1	

All except two of the columns are obscured, and while it is clear that there are five rows you cannot read anything of the other columns, including their position. Prove that there is an error in the table. Given that there is exactly *one* error, determine where it is, and what the correct entry should be.

8. A finite group has seven conjugacy classes of elements with representatives $c_1, \ldots, c_7$ (where $c_1 = 1$), and the values of five of its irreducible characters are given by the following table:

c_1	c_2	c_3	c_4	c_5	c_6	c_7
1	1	1	1	1	1	1
1	1	1	1	-1	-1	-1
4	1	-1	0	2	-1	0
4	1	-1	0	-2	1	0
5	-1	0	1	1	1	-1

Calculate the numbers of elements in the various conjugacy classes and the remaining simple characters.

Extra questions: do not upload to Gradescope

9. Let $g \in G$.

(a) Prove that g lies in the center of G if and only if $|\chi(g)| = |\chi(1)|$ for every simple complex character χ of G.

(b) Show that if G has a faithful simple complex character (one whose kernel is 1) then the center of G is cyclic. (You may assume that every finite subgroup of $\mathbb{C}$ is cyclic.)

10. Let U be a module for a semisimple finite dimensional algebra A. Show that if $\mathrm{End}_A(U)$ is a division ring then U is simple.

11.(a) By using characters show that if V and W are $\mathbb{C}G$-modules then $(V \otimes_{\mathbb{C}} W)^* \cong V^* \otimes_{\mathbb{C}} W^*$, and $({}_{\mathbb{C}G}\mathbb{C}G)^* \cong {}_{\mathbb{C}G}\mathbb{C}G$ as $\mathbb{C}G$-modules.

(b) If k is any field and V, W are kG-modules, show that $(V \otimes_k W)^* \cong V^* \otimes_k W^*$, and $({}_{kG}kG)^* \cong {}_{kG}kG$ as kG-modules. (Can you guess maps that might be isomorphisms?)

12. Here is a column of a character table:

$$\begin{array}{c} g \\ \hline 1 \\ -1 \\ 0 \\ -1 \\ -1 \\ \frac{-1+i\sqrt{11}}{2} \\ \frac{-1-i\sqrt{11}}{2} \\ 0 \\ 1 \\ 0 \end{array}$$

(a) Find the order of g.

(b) Prove that $g \notin Z(G)$.

(c) Show that there exists an element $h \in G$ with the same order as g but not conjugate to g.

(d) Show that there exist two distinct simple characters of G of the same degree.