Math 8300 **Homework 3** **due Saturday 11/27/2021**
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1. Let G be the non-abelian group of order 21:

$$G = \langle x, y \mid x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle.$$

Show that G has 5 conjugacy classes, and find its character table. [The answer and a brief hint are given in the character tables section at the end of my text book. If you look there, make sure to present the calculations that are suggested.]

2. Let H and K be subgroups of G with $HK = G$ and $H \cap K = 1$. Show that for any kH-module U the module $U \uparrow_H^G \downarrow_K^G$ is a direct sum of copies of the regular representation kK.

3. Let k be a field. Show by example that it is possible to find a subgroup H of a group G and a simple kG-module U for which $U \downarrow_H^G$ is not semisimple.

4. Let H be a subgroup of G and V an RH-module. Show that if V can be generated by d elements as an RH-module then $V \uparrow_H^G$ can be generated by d elements as an RG-module.

5. Let H be a subgroup of G.
(a) Write $\overline{H} = \sum_{h \in H} h$ for the sum of the elements of H, as an element of RG. Show that $RG \cdot \overline{H} \cong R \uparrow_H^G$ as left RG-modules. Show also that $RG \cdot \overline{H}$ equals the fixed points of H in its action on RG from the *right*.
(b) More generally let $\rho : H \to R^\times$ be a 1-dimensional representation of H (that is, a group homomorphism to the units of R). Write $\tilde{H} := \sum_{h \in H} \rho(h)h \in RG$. Show that $RG \cdot \tilde{H} \cong \rho^* \uparrow_H^G$ as RG-modules.

6. Let k be any field, and g any element of a finite group G.
(a) If $K \leq H \leq G$ are subgroups of G, V a kH-module, and W a kK-module, show that $({}^gV) \downarrow_{{}^gK}^{{}^gH} \cong {}^g(V \downarrow_K^H)$ and $({}^gW) \uparrow_{{}^gK}^{{}^gH} \cong {}^g(W \uparrow_K^H)$. [This allows us to put conjugation before, between, or after restriction and induction in Mackey's formula.]
(b) If U is any kG-module, show that $U \cong {}^gU$ by showing that one of the two mappings $U \to {}^gU$ specified by $u \mapsto gu$ and $u \mapsto g^{-1}u$ is always an RG-module isomorphism. [Find which one of these it is.]

7. (Artin's Induction Theorem) Let $\mathbb{C}^{\mathrm{cc}(G)}$ denote the vector space of class functions on G and let $\mathcal{C}$ be a set of subgroups of G that contains a representative of each conjugacy class of cyclic subgroups of G. Consider the linear mappings

$$\mathrm{res}_{\mathcal{C}} : \mathbb{C}^{\mathrm{cc}(G)} \to \bigoplus_{H \in \mathcal{C}} \mathbb{C}^{\mathrm{cc}(H)}$$

and

$$\mathrm{ind}_{\mathcal{C}} : \bigoplus_{H \in \mathcal{C}} \mathbb{C}^{\mathrm{cc}(H)} \to \mathbb{C}^{\mathrm{cc}(G)}$$

whose component homomorphisms are the linear mappings given by restriction

$$\downarrow_H^G : \mathbb{C}^{\mathrm{cc}(G)} \to \mathbb{C}^{\mathrm{cc}(H)}$$

and induction

$$\uparrow_H^G : \mathbb{C}^{\mathrm{cc}(H)} \to \mathbb{C}^{\mathrm{cc}(G)}$$

(a) With respect to the usual inner product $\langle \quad , \quad \rangle_G$ on $\mathbb{C}^{\mathrm{cc}(G)}$ and the inner product on $\bigoplus_{H\in\mathcal{C}} \mathbb{C}^{\mathrm{cc}(H)}$ that is the orthogonal sum of the $\langle \quad , \quad \rangle_H$, show that $\mathrm{res}_{\mathcal{C}}$ and $\mathrm{ind}_{\mathcal{C}}$ are the transpose of each other. [For this we have to realize that if $\alpha : V \to W$ is a linear map then the usual transpose of α is a linear map $\beta : W \to V$ satisfying $\langle \alpha(v), w\rangle_W = \langle v, \beta(w)\rangle_V$ where $\langle -, -\rangle_V$ and $\langle -, -\rangle_W$ are the standard inner products on V and W defined with respect to given bases of V and W. A transpose may be defined using any pair of inner products like this.]
(b) Show that $\mathrm{res}_{\mathcal{C}}$ is injective.
[Use the fact that $\mathbb{C}^{\mathrm{cc}(G)}$ has a basis consisting of characters, that take their information from cyclic subgroups.]
(c) Prove Artin's induction theorem: In $\mathbb{C}^{\mathrm{cc}(G)}$ every character χ can be written as a rational linear combination

$$\chi = \sum a_{H,\psi}\psi \uparrow_H^G$$

where the sum is taken over cyclic subgroups H of G, ψ ranges over characters of H and $a_{H,\psi} \in \mathbb{Q}$.
[Deduce this from surjectivity of $\mathrm{ind}_{\mathcal{C}}$ and the fact that it is given by a matrix with integer entries. A stronger version of Artin's theorem is possible: there is a proof due to Brauer which gives an explicit formula for the coefficients $a_{H,\psi}$; from this we may deduce that when χ is the character of a $\mathbb{Q}G$-module the ψ that arise may all be taken to be the trivial character.]
(d) Show that if U is any $\mathbb{C}G$-module then there are $\mathbb{C}G$-modules P and Q, each a direct sum of modules of the form $V \uparrow_H^G$ where H is cyclic, for various V and H, so that $U^n \oplus P \cong Q$ for some n, where U^n is the direct sum of n copies of U.

Extra questions: do not upload to Gradescope

8. Find the character table of the following group of order 36:

$$G = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^2\rangle.$$

[It follows from these relations that $\langle a, b\rangle$ is a normal subgroup of G of order 9.]

9. Compute the character table of the symmetric group S_5 by the methods we have seen. It can all be done by considering the decomposition of permutation representations, and tensor product with the sign representation is useful.

10. Compute the character tables of the alternating groups A_4 and A_5 using the following procedure. You may assume that A_5 is a simple group that is isomorphic to the group of rotations of a regular icosahedron, and that A_4 is isomorphic to the group of rotations of a regular tetrahedron.
(a) Compute the conjugacy classes by observing that each conjugacy class of even permutations in S_n is either a single class in A_n or the union of two classes of A_n, and that this can be determined by computing centralizers of elements in A_n and comparing them with the centralizers in S_n.
(b) Compute the abelianization of each group, and hence the 1-dimensional representations.
(c) Obtain further representations using the methods of this section. We have natural 3-dimensional representations in each case. It is also helpful to consider induced representations from the Sylow 2-subgroup in the case of A_4, and from the subgroup A_4 in the case of A_5.

11. Find the complete list of subgroups H of the dihedral group D_8 such that the 2-dimensional simple representation over $\mathbb{C}$ can be written $U \uparrow_H^G$ for some 1-dimensional representation U of H. Do the same thing for the quaternion group Q_8.

12. We saw the character table of the semidihedral group of order 16 in class:

$$SD_{16} = \langle x, y \mid x^8 = y^2 = 1,\ yxy^{-1} = x^3 \rangle.$$

Compute the character table of the generalized quaternion group of order 16

$$Q_{16} = \langle x, y \mid x^8 = 1,\ x^4 = y^2,\ yxy^{-1} = x^{-1} \rangle$$

13. The following statements generalize Maschke's theorem. Let H be a subgroup of G and suppose that k is a field in which $|G : H|$ is invertible. Let V be a kG-module.
(a) Show that

$$\frac{1}{|G : H|} \sum_{g \in [G/H]} g : V^H \to V^G$$

is a well-defined map that is a projection of the H-fixed points onto the G-fixed points. In particular, this map is surjective.
(b) Show that if $V \downarrow_H^G$ is semisimple as a kH-module then V is semisimple as a kG-module.

14. Let H be a normal subgroup of G and suppose that k is a field of characteristic p.
(a) Let $p \nmid |G : H|$. Show that if U is a semisimple kH-module then $U \uparrow_H^G$ is a semisimple kG-module.
(b) Let $p \mid |G : H|$. Show by example that if U is a semisimple kH-module then it need not be the case that $U \uparrow_H^G$ is a semisimple kG-module.

15. Let H be a subgroup of G of index 2 (so that H is normal in G) and let k be a field whose characteristic is not 2. The homomorphism $G \to \{\pm 1\} \subset k$ with kernel H is a

1-dimensional representation of G that we will call ϵ. Let S, T be simple kG-modules and let U, V be simple kH-modules. You may assume that $U \uparrow_H^G$ and $V \uparrow_H^G$ are semisimple (proved in a different exercise). Let $g \in G - H$.
(a) Show that $S \downarrow_H^G$ is the direct sum of either 1 or 2 simple kH-modules.
(b) Show that $U \uparrow_H^G$ is the direct sum of either 1 or 2 simple kG-modules.
In the following questions, notice that

$$S \downarrow_H^G \uparrow_H^G \cong S \otimes (k \uparrow_H^G) \cong S \otimes (k \oplus \epsilon) \cong S \oplus (S \otimes \epsilon).$$

For some parts of the questions it may help to consider

$$\mathrm{Hom}_{kH}(S \downarrow_H^G, T \downarrow_H^G) \quad \text{and} \quad \mathrm{Hom}_{kG}(U \uparrow_H^G, V \uparrow_H^G).$$

(c) Show that the following are equivalent:
(i) S is the induction to G of a kH-module,
(ii) $S \downarrow_H^G$ is not simple,
(iii) $S \cong S \otimes \epsilon$.
(d) Show that the following are equivalent:
(i) U is the restriction to H of a kG-module,
(ii) $U \uparrow_H^G$ is not simple,
(iii) $U \cong {}^gU$.
(e) Show that $S \downarrow_H^G$ and $T \downarrow_H^G$ have a summand in common if and only if $S \cong T$ or $S \cong T \otimes \epsilon$.
(f) Show that $U \uparrow_H^G$ and $V \uparrow_H^G$ have a summand in common if and only if $U \cong V$ or $U \cong {}^gV$.
(g) We place an equivalence relation $\sim_1$ on the simple kG-modules and an equivalence relation $\sim_2$ on the simple kH-modules:

$$S \sim_1 T \Leftrightarrow S \cong T \text{ or } S \cong T \otimes \epsilon$$
$$U \sim_2 V \Leftrightarrow U \cong V \text{ or } U \cong {}^gV.$$

Show that induction $\uparrow_H^G$ and restriction $\downarrow_H^G$ induce mutually inverse bijections between the equivalence classes of simple kG-modules and of simple kH-modules in such a way that an equivalence class of size 1 corresponds to an equivalence class of size 2, and vice-versa.
(h) Show that the simple kG-modules of odd degree restrict to simple kH-modules, and the number of such modules is even.
(i) In the case where $G = S_4$, $H = A_4$ and $k = \mathbb{C}$, show that there are three equivalence classes of simple characters under $\sim_1$ and $\sim_2$. Verify that $\downarrow_{A_4}^{S_4}$ and $\uparrow_{A_4}^{S_4}$ give mutually inverse bijections between the equivalence classes.

16. Show that every simple representation of $C_3 \times C_3$ over $\mathbb{R}$ has dimension 1 or 2. Deduce that if V is a simple 2-dimensional representation of C_3 over $\mathbb{R}$ then $V \otimes V$ is not a simple $\mathbb{R}[C_3 \times C_3]$-module.