Math 8300 **Homework 4** **due Saturday 12/18/2021**
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Questions with more parts count for more.

1. Let $G = C_2 \times C_2$ be the Klein four group with generators a and b, and $k = \mathbb{F}_2$ the field of two elements. Let V be a 3-dimensional space on which a and b act via the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Determine (with an argument) whether or not this representation is indecomposable.
(b) Draw a diagram to represent this module, where the nodes in the diagram biject with the vectors in a basis for this module, and there are arrows between the nodes corresponding to the action of $a - 1$ and $b - 1$.

2. (a) Prove that if N is a normal subgroup of G and k is a field then $\mathrm{Rad}(kN) = kN \cap \mathrm{Rad}(kG)$. (Consider using Clifford's theorem and the various things we know about the radical.)
(b) Show by example that if H is a subgroup of G that is not normal then it need not be true that $\mathrm{Rad}\, kH \subseteq \mathrm{Rad}\, kG$ (in which case $\mathrm{Rad}(kH) \neq kH \cap \mathrm{Rad}(kG)$).

3. Show that the following conditions are equivalent for a module U that has a composition series.
(a) U is uniserial (i.e. U has a unique composition series).
(b) The set of all submodules of U is totally ordered by inclusion.
(c) $\mathrm{Rad}^r U / \mathrm{Rad}^{r+1} U$ is simple for all r.
(d) $\mathrm{Soc}^{r+1} U / \mathrm{Soc}^r U$ is simple for all r.

4. Let A be a finite dimensional algebra over a field. Show that A is semisimple if and only if all finite dimensional A-modules are projective.

5. (a) Show that $\mathbb{F}_3 S_3$ has two isomorphism classes of simple modules.
(b) Let $e_1 \in \mathbb{F}_3 S_3$ be the idempotent $e_1 = \frac{1}{2}(() + (1,2))$, let $e_{-1} = \frac{1}{2}(() - (1,2))$, and consider the direct sum decomposition of left $\mathbb{F}_3 S_3$-modules $\mathbb{F}_3 S_3 = \mathbb{F}_3 S_3 e_1 \oplus \mathbb{F}_3 S_3 e_{-1}$. Show that, on restriction to the cyclic subgroup $\langle (1,2,3) \rangle$, each of the two modules in this direct sum is a copy of $\mathbb{F}_3 \langle (1,2,3) \rangle$. Deduce that each module is indecomposable and uniserial as an $\mathbb{F}_3 S_3$-module.
(c) By considering a basis of each of these two indecomposable modules compatible with the action of $(1,2,3) - ()$ (or otherwise) and the action of (12) on this basis, identify the isomorphism types of the composition factors of these indecomposable modules, showing that the Cartan matrix of $\mathbb{F}_3 S_3$ is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

6. The setup in this question is that U, V are (finite dimensional) kG-modules where k is a field. We write $U^* = \mathrm{Hom}_k(U, k)$ for the dual kG-module to U. We suppose we are given a non-degenerate bilinear pairing

$$\langle \quad , \quad \rangle : U \times V \to k$$

which has the property $\langle u, v\rangle = \langle gu, gv\rangle$ for all $u \in U$, $v \in V$, $g \in G$. (A *pairing* is like a bilinear form, except the spaces U and V may be different spaces. Non-degenerate means that the matrix of the pairing is non-degenerate, just like with bilinear forms, and there are other ways to express this, such as the left and right kernels are zero.) If U_1 is a subspace of U let $U_1^\perp = \{v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U_1\}$ and if V_1 is a subspace of V let $V_1^\perp = \{u \in U \mid \langle u, v\rangle = 0 \text{ for all } v \in V_1\}$.
(a) Show that if U_1 and V_1 are kG-submodules, then so are $U_1^\perp$ and $V_1^\perp$.
(b) Show that the mapping $v \mapsto (u \mapsto \langle u, v\rangle)$ is an isomorphism $V \cong U^*$ as kG-modules.
(c) Show that if U_1 and U_2 are kG-submodules of U then $U_1 \subseteq U_2$ if and only if $U_1^\perp \supseteq U_2^\perp$. Show further in this case that

$$U_1^\perp / U_2^\perp \cong (U_2/U_1)^*$$

as kG-modules. Notice (but do not write anything about it) that the lattice of submodules of U is the opposite of the lattice of submodules of U^*.
(d) Let G permute a set Ω and let $U = V = k\Omega$ be the permutation module. Define $\langle \quad , \quad \rangle$ on basis elements $u, v \in \Omega$ by $\langle u, v\rangle = \delta_{u,v}$ (the Kronecker delta). Show that this pairing satisfies the condition $\langle u, v\rangle = \langle gu, gv\rangle$ always. Deduce that $k\Omega \cong (k\Omega)^*$. Deduce that if all indecomposable summands of $k\Omega$ have simple radical quotients, then they also all have simple socles.

Extra questions: do not upload to Gradescope

7. (a) Over any coefficient ring. R, show that if N is a normal subgroup of G then the left ideal

$$RG \cdot IN = \{x \cdot y \mid x \in RG, y \in IN\}$$

of RG generated by IN is the kernel of the ring homomorphism $RG \to R[G/N]$ and is in fact a 2-sided ideal in RG.
[One approach to this uses the formula $g(n - 1) = ({}^g n - 1)g$.]
Show further that $(RG \cdot IN)^r = RG \cdot (IN)^r$ for all r.
(b) Now let k be a field of characteristic p and suppose that G has a normal Sylow p-subgroup N. Show that $\mathrm{Rad}\, kG = kG \cdot \mathrm{Rad}\, kN$.
[Use what you know about the radical, showing that $k[G/N]$ is the largest semisimple quotient of kG.]

8. Let D_{30} be the dihedral group of order 30.

(a) By using the fact that D_{30} has a normal Sylow 5-subgroup with quotient $S_3 \cong D_6$, show that $\mathbb{F}_5 D_{30}$ has three simple modules of dimensions 1, 1 and 2. We will label them k_1, k_ϵ and U, respectively, with k_1 the trivial module.
(b) Show that the indecomposable projectives P_{k_1} and P_{k_ϵ} each have dimension 5, that P_2 has dimension 10, and $\mathbb{F}_5 D_{30} \cong P_{k_1} \oplus P_{k_\epsilon} \oplus P_2 \oplus P_2$. Show that each indecomposable projective is uniserial with composition series of length 5.